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DEPARTMENT OF PHYSICS AND ASTRONOMY — 2355 Sixth Str., NW
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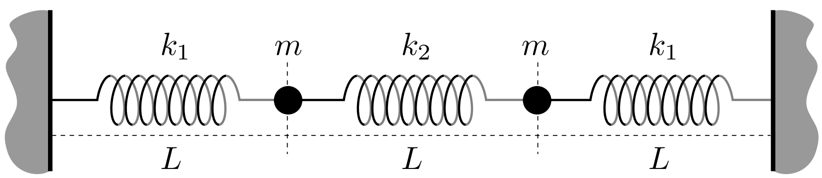
— Classical Mechanics —

Work four problems only. If you work on more than four problems, you *must* identify which four are to be graded.

1. Examine two equal masses attached to springs and constrained to move in one (horizontal) dimension, as shown below:

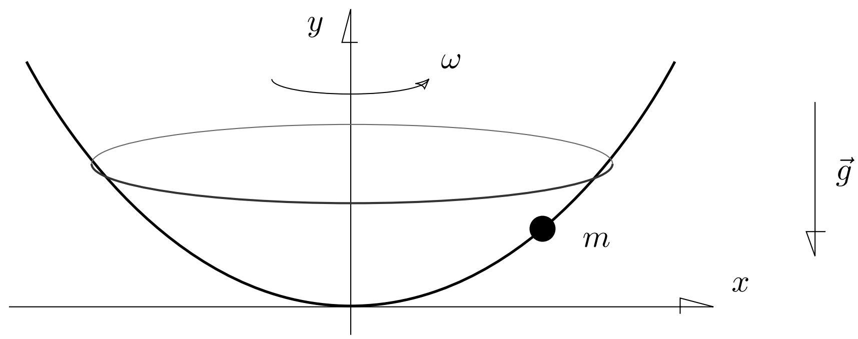


a. Determine the frequencies of the normal modes for the system.

b. Obtain the normalized eigenvectors for the motion of the masses.

c. Next, the left mass is displaced by $L/2$ towards the right one which is held stationary at its equilibrium position, and both masses are then released from rest at $t = 0$. Determine the subsequent motion of the masses.

2. A bead slides without friction on a rotating parabolic wire (the shape of which is described by $y = ax^2$) as shown below:



a. Determine the stable equilibrium points for the bead for all values of ω.

b. Obtain the equation of motion for small oscillations about the equilibrium points determined above.

c. Determine the frequencies for small oscillations about the equilibrium points.

3. Consider the motion of a heavy axially symmetric top, in a uniform gravitational field, with one fixed point which lies on the axis of symmetry. Neglect friction. The components of angular velocity $\vec{\omega}$ in terms of the Euler angles are given by:

$$\omega_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \ ,$$

$$\omega_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \ ,$$

$$\omega_3 = \dot{\phi} \cos\theta + \dot{\psi} \ .$$

a. Draw a diagram, clearly showing the Euler angles (θ, ϕ, ψ).

b. Define the principal axes, moments of inertia (I_1, I_2, I_3), and use symmetry arguments to prove that two of the I_i's are equal.

c. Write down the Lagrangian of the system.

d. Which coordinates are cyclic? What are the conserved quantities? Give reasons.

e. Write expressions for the momenta conjugate to the cyclic coordinates and for the total energy of the system.

4. The science-fiction writer R.A. Heinlein describes the "skyhook" geostationary satellite to consist, in its simplest form, of a cable of uniform mass per length (ρ), placed in the equatorial plane, oriented radially and extending from just above the surface of the planet to a length L above the planet.

a. Write down the condition for "skyhook" to be in equilibrium by considering an infinitesimal segment of the cable.

b. Find the required length of "skyhook" for it to orbit around the Earth.

c. Determine the height at which a conventional geostationary satellite has to be positioned, and compare that with the length of the "skyhook".

d. Consider finally a conventional geostationary satellite (in the equatorial plane) of mass m which supports a cable of mass per unit length ρ, hanging down to the Earth surface. Find the equation determining the height at which such a satellite has to orbit and show that it depends only on the ratio m/ρ, not on m and ρ separately. (You need not solve this equation.)

The radius of Earth is $R_E = 6.4\times10^6$ m; the gravitational acceleration at the surface is $9.81\,\mathrm{m/s^2}$.

5. A heavy particle of mass m, placed at the top of a vertical hoop of radius a and initially at rest, slides on the hoop.

a. Using polar coordinates, write the Lagrangian for the motion of the particle along the hoop.

b. Write the equation(s) of constraint for the particle. Using Lagrange multiplier(s), obtain the Lagrange equations of motion.

c. Calculate the reaction force of the hoop on the particle.

d. Find the height at which the particle falls off the hoop.

6. Consider a one-dimensional harmonic oscillator of mass m and spring constant k.

a. Write the Hamiltonian in terms of the canonical coordinate q and the conjugate momentum p.

b. Using the Poisson bracket $\{Q_i, P_j\} = \delta_{ij}$, find the condition necessary for a change of variables $(q, p) \to (Q, P)$ to be a canonical transformation.

c. Using the results from part b., determine the value of the constant C so that the equations

$$Q = C(p + im\omega q) \quad \text{and} \quad P = C(p - im\omega q) \ , \qquad \omega = \sqrt{k/m} \ ,$$

define a canonical transformation.

d. Determine the generating function $S(q, P)$ for the canonical transformation in part c.

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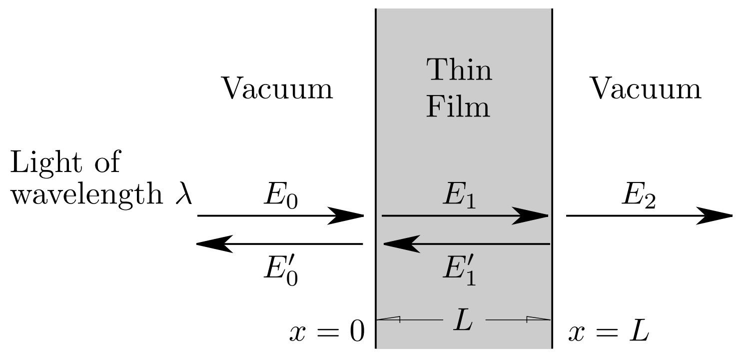
— Electromagnetism —

Work four problems only. If you work on more than four problems, you *must* identify which four are to be graded.

1. Light of wavelength λ is incident normally on a thin film of thickness L and refraction index n:



a. What are the boundary conditions at $x = 0$ and $x = L$?

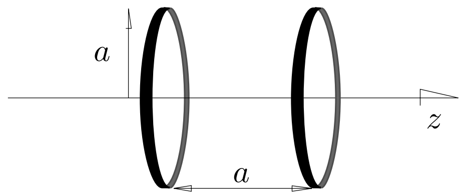
b. What is the reflectance $R = |E_0'/E_0|^2$?

c. What are the conditions for maximum R (constructive interference) and for minimum R (destructive interference)?

d. What is the maximum reflectivity?

(You may assume $\vec{B} = \vec{H}$ everywhere; $\vec{B} = \mu_0\vec{H}$ in MKS system.)

2. Helmholtz coils are designed to produce uniform magnetic fields. Two circular coils, each of N turns and with the same radius a are separated by the distance a along the common axis, z. Both coils carry the same current I in the same direction. At the midpoint. on the z axis, between the coil centers



a. Determine the magnetic field $\vec{B}$.

b. Show that at this point,

$$\frac{\partial B_z}{\partial z} = 0 \ , \qquad \frac{\partial^2 B_z}{\partial z^2} = 0 \ , \qquad \frac{\partial^3 B_z}{\partial z^3} = 0 \ .$$

c. In a general physics laboratory, it is desired to produce a uniform magnetic field of 5×10^{-4} T using Helmholtz coils with $N = 65$ (for each coil), and $a = 0.15$ m. What is the required current I (in amperes)?

($\mu_0 = 4\pi\times10^{-7}$ H/m)

3. Consider a large parallel plate capacitor constructed of two circular plates of radius R each, separated by $d \ll R$, and charged with a time-dependent charge $Q(t)$.

a. Determine the electric field and the magnetic field to lowest order for all $r \ll R$ inside the capacitor.

b. Neglecting edge effects, determine the electromagnetic power (energy flux) through a cylinder at the edge, $r = R$, of the capacitor.

c. Determine the force of one plate on the other, to the lowest order, using Maxwell's stress tensor.

d. Determine the next order correction, due to induction, to the electric field within the capacitor. Determine a criterion for neglecting this higher order correction, as compared to the result obtained in part a.

4. An electron at rest is released from a large (but finite) distance toward a nucleus of charge Ze, and thus "falls" toward the nucleus.

a. Calculate the angular distribution of the emitted radiation.

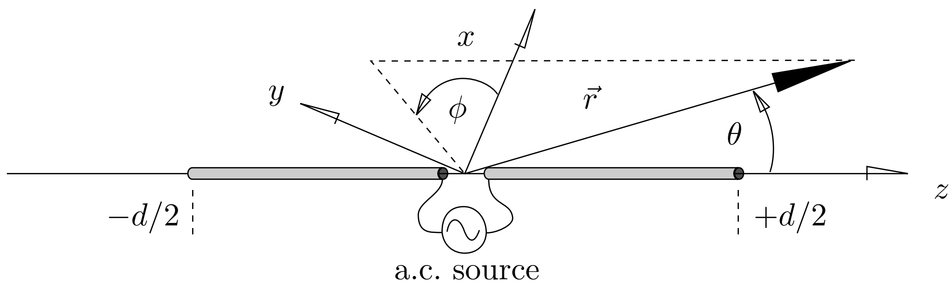
b. Calculate the polarization of the emitted radiation.

c. Calculate the radiated power as a function of the separation between the electron and the nucleus.

d. Calculate the total energy radiated between distances r_1 and r_2 (with $r_2 < r_1$).

Assume that $v \ll c$ at all times and neglect the radiative reaction force on the electron.

5. Consider the radiating system consisting of two short straight pieces of wire connected to an alternating current source, as shown in the diagram below. The source excites each wire across the gap between them, such that each wire carries a symmetric sinusoidal current of wave number $k = \omega/c$. The current is zero at the ends of the wire and has a peak value I.



a. Determine the vector potential at some point (r, θ, ϕ) in the space around the wire, where $r > \lambda > d$, but θ and ϕ are arbitrary.

b. Determine the magnetic induction, $\vec{B}$, in the radiation zone.

b. Determine the time-averaged power radiated per unit solid angle.

6. Examine a dielectric sphere, of radius a and dielectric constant ϵ_1, embedded in another dielectric medium (ϵ_2) with an asymptotically homogeneous electric field E_0 oriented parallel to the z-axis.

a. Determine the potential throughout all space.

b. Determine the bound surface charge density at $r = a$.

c. Suppose now that a long wavelength electromagnetic wave of amplitude E_0 and frequency ω polarized in the $\hat{z}$ direction is incident on the sphere traveling parallel to the x-axis. Demonstrate the form of the incident wave.

d. Determine the asymptotic form of the electromagnetic radiation generated by the dielectric sphere. It is sufficient to determine the form of the scalar and vector potentials.

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— Quantum Mechanics —

Work four problems only. If you work on more than four problems, you *must* identify which four are to be graded.

1. Consider two operators, $\hat{A}$ and $\hat{B}$ which satisfy:

$$\left[\hat{A}, \hat{B}\right] = \hat{B} , \qquad \hat{B}^{\dagger}\hat{B} = \mathbb{1} - \hat{A}^2 , \qquad \hat{A}\,|a\rangle = a\,|a\rangle \ .$$

a. Determine the hermiticity properties of $\hat{A}$ and $\hat{B}$.

b. Using the fact that $|a = 0\rangle$ is an eigenstate of $\hat{A}$, construct the other eigenstates of $\hat{A}$.

c. Suppose the eigenstates of $\hat{A}$ form a complete set. Determine if eigenstates of $\hat{B}$ can be constructed, and if so, determine the spectrum of the eigenstates of $\hat{B}$.

2. Consider a 2-dimensional harmonic oscillator, for which the Hamiltonian can be written as $H = \frac{1}{2m}({p_x}^2 + {p_y}^2) + \frac{1}{2}m\omega^2(x^2 + y^2)$.

a. Write down energies of the allowed states of this oscillator (in units of $\hbar\omega$) and specify their degeneracy.

b. For a suitable constant α, does a perturbation of the form $V = \alpha x$ change the degeneracy? Why (why not)?

c. For a suitable constant β, does a perturbation of the form $V = \beta x^2$ change the degeneracy? Why (why not)?

d. For a suitable constant γ, does a perturbation of the form $V = \gamma x^4$ change the degeneracy? Why (why not)?

e. Use perturbation theory to calculate the first order shift in the ground state energy, caused by a small perturbation $V = \gamma x^4$.

f. For all of the above perturbations and for any arbitrary collection of states, is it necessary to use *degenerate* perturbation theory? Why (why not)?

3. A particle of mass m moves in 3-dimensional space under the influence of the ("opaque bubble") potential of the form $V(r) = -\gamma\,\delta(r-a)$, for a, γ positive constants.

a. Describe the general form of the spectrum. For which values of the energy is the spectrum discrete, and for which values is it continuous?

b. Write down the Schrödinger equation in spherical coordinates, and obtain the radial equation for $u_{E,\ell}(r)$, assuming that the wave function can be written as $\Psi(r,\theta,\phi) = r^{-1}\,u_{E,\ell}(r)\,Y_\ell^m(\theta,\varphi)$, where $Y_\ell^m(\theta,\varphi)$ are the spherical harmonics.

c. Describe the S-wave solutions (ℓ=?). Sketch their radial function $u_{E,\ell}(r)$, and specify all the boundary/matching conditions.

d. From the boundary/matching conditions, find the transcendental equation which determines the energy quantization (for the discrete part of the spectrum), for $\ell = 0$.

e. Determine the "translucent" limit, *i.e.*, the smallest value of γ for which there is precisely one bound state. Find a lowest order estimate for the energy of this state.

(In spherical coordinates, $\nabla^2\Psi = \frac{\partial^2\Psi}{\partial r^2} + \frac{2}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\varphi^2}$.)

4. Examine the pionic decays of K^0 governed by weak interactions. The dominant decays are $K^0 \to \pi^+\pi^-$ and $K^0 \to \pi^0\pi^0$. Total isospin is *not* conserved in these processes, but changes either by $\triangle I = +\frac{1}{2}$ or by $\triangle I = -\frac{1}{2}$.

a. Introducing appropriate creation and annihilation operators, write down the interaction terms ($\hat{H}_I$) in the Hamiltonian corresponding to the given decay processes. Given the basic particle data for K^0 and $\pi^0, \pi^\pm$ (given below), does $\hat{H}_I$ commute with isospin vector-operator $\hat{\vec{I}}$? Why (why not)?

b. Using the basic particle data (below), determine the possible values of the angular momentum and isospin of each of the two-pion system into which the K^0 decays.

c. Based only on the particle data below, estimate the ratio of cross-sections $\frac{\sigma(K^0 \to \pi^+\pi^-)}{\sigma(K^0 \to \pi^0\pi^0)}$.

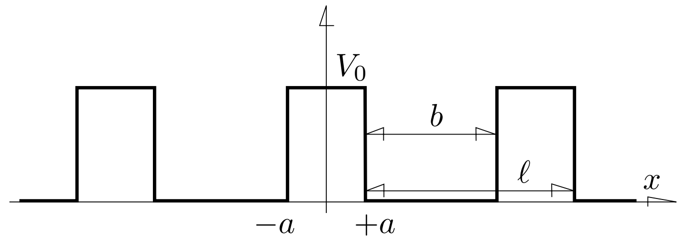
d. If the charge-conjugation & parity (reflection) operator acts as $\mathrm{CP}\big|K^0\big\rangle = |\overline{K^0}\rangle$ and obeys $(\mathrm{CP})^2 = \mathbb{1}$, find the ortho-normalized kaon (K) CP-eigenstates.

e. For $|K_\pm\rangle$, such that $\mathrm{CP}|K_\pm\rangle = \pm|K_\pm\rangle$, let $\Gamma_\pm$ denote the decay rate. What is the fraction of K^0's in an initially pure K^0-beam, as a function of proper time?

f. Is the decay $K^0 \to \pi^+\pi^0\pi^-$ possible? Is the decay $K^0 \to \pi^+\pi^0\pi^0\pi^-$ possible? Prove your assertions by a short calculation.

The mesons, $K^0, \pi^\pm, \pi^0$, have no spin and are odd under parity (space reflection). The $K^0, \overline{K^0}$ have (isospin) $I_z = -\frac{1}{2}, +\frac{1}{2}$, respectively, while the π^+, π^0, π^- have $I_z = +1, 0, -1$, respectively. The (rest-) masses are (in MeV/c^2): $m_{K^0} = 497.7$, $m_{\pi^0} = 135.0$, $m_{\pi^\pm} = 139.6$. Some possibly useful Clebsh-Gordan coefficients: $\langle 1,1;1,-1|2,0\rangle = 1/\sqrt{6}$, $\langle 1,1;1,-1|1,0\rangle = 1/\sqrt{2}$, $\langle 1,1;1,-1|0,0\rangle = 1/\sqrt{3}$, $\langle 1,1;0,0|2,0\rangle = \sqrt{2/3}$, $\langle 1,1;0,0|1,0\rangle = 0$, $\langle 1,1;0,0|0,0\rangle = -1/\sqrt{3}$.

5. Consider a particle of mass μ in a 1-dimensional periodic potential shown in the figure. The height of the barriers is V_0, and the potential satisfies $V(x+\ell) = V(x)$:



a. Using the translational symmetry, prove that there is a complete set of stationary states which obey
$$\psi_{_E}(x+\ell) = e^{iK\ell}\psi_{_E}(x)\ , \qquad E > 0\ ,$$
where K is a constant.

b. Determine this K by imposing a periodic boundary condition on the wave function over a large but finite region with N barriers: $\psi_{_E}(x+N\ell) \equiv \psi_{_E}(x)$, for all E.

c. Write down the general solution within $-a \leq x \leq \ell+a$, and for any $E > 0$. Using the results from parts a. and b., reduce the number of undetermined constants to four. Then use the boundary matching conditions to find the system of equations which determins the remaining four constants (you need not solve this system).

d. When $0 < E < V_0$, and writing $k\hbar = \sqrt{2mE}$ and $\kappa\hbar = \sqrt{2m(V_0 - E)}$, the energy condition
$$\cos(kb)\cosh(2\kappa a) + \frac{\kappa^2 - k^2}{2\kappa k}\sin(kb)\sinh(2\kappa a) = \cos(K\ell)$$
must be enforced for $\psi_{_E} \not\equiv 0$. Show that this forbids certain regions of energy.

e. Considering carefully the limit when $a \to 0$ and $V_0 \to \infty$ but $2aV_0 = \Omega = \textit{constant}$, find the resulting energy condition and obtain the lowest order estimate for the minimum allowed energy.

6. The spin Hamiltonian for a spin-$\frac{1}{2}$ particle in a magnetic field is given by $H = \vec{\omega}\cdot\vec{S}$, where the matrix representation of the spin operators is given by $[\vec{S}] = \frac{1}{2}\hbar\vec{\boldsymbol{\sigma}}$, in terms of the usual Pauli matrices $\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2, \boldsymbol{\sigma}^3$.

a. Show that the time evolution operator for the quantum state vectors takes the form $\boldsymbol{U}(t) = e^{i\mathbf{M}t}$, where $\mathbf{M}^2 = \mathbb{1}$.

b. Expand the time evolution matrix to show that it is proportional to a linear combination of $\mathbb{1}$ and $\mathbf{M}$.

c. If the system is in the state $\left|m = +\frac{1}{2}\right\rangle$ at time $t = 0$, determine the state of the system at a later time $t > 0$.

d. Determine the probability that the system is measured to be in state $\left|m = +\frac{1}{2}\right\rangle$ at a later time $t > 0$.

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— Statistical Mechanics —

Work four problems only. If you work on more than four problems, you *must* identify which four are to be graded.

1. A chunk of graphite is in equilibrium with argon gas at pressure p and at absolute temperature T. The graphite has N_a sites on its surface, each of which can absorb one argon atom. The energy decrease for the absorption of each atom is ϵ. Assume that the argon gas is ideal and that the number of the argon atoms, N, is much larger than the number of the absorption sites, N_a (*i.e.*, $N \gg N_a$).

a. Calculate the free energy of argon.

b. Calculate the chemical potential of argon.

c. Give the grand canonical partition function for the absorbed argon atoms. Neglect the interaction among absorption sites.

d. Compute the fraction of occupied sites in terms of the temperature, T, and the gas pressure p.

e. Sketch the fraction of the occupied sites as a function of the gas pressure.

2.a. Show that the first law of thermodynamics can be written as

$$\mathrm{d}U \;=\; C_V \mathrm{d}T \;+\; \left[T \left(\frac{\partial p}{\partial T} \right)_V - p \right] \mathrm{d}V$$

Suppose that C_V is independent of T, and that the equation of state is

$$p = \chi \big(T \log T - T \big) \left(\frac{N}{V} \right)^{\gamma} \, .$$

b. Determine the entropy of the system.

c. Determine the heat capacity and the energy of the system.

d. Determine the energy change in the system during an adiabatic expansion from volume V_0 to V_f; you may leave your answer in integral form.

3. Consider a system of N hydrogen atoms in a volume V, at temperature T, such that a significant number of the atoms are ionized. We shall find a formula which enables the calculation of the ratio of ionized to unionized atoms.

a. Calculate the free energy of a non-interacting classical gas composed of equal number of electrons and protons.

b. Calculate the free energy of a classical gas of non-interacting hydrogen atoms, all of which are in the ground state E_0.

c. Assume that a non-interacting mixture of hydrogen atoms, protons and electrons is formed by partial ionization of the hydrogen at a temperature T. Minimize the free energy with respect to the number of atoms in the gas under the condition that the total number of hydrogen atoms was initially (before ionization) equal to N, and that the gas is always neutral (net charge is zero). Obtain a relation between the number of hydrogen atoms and the number of electrons in equilibrium at temperature T.

d. Suppose all the H atom states are occupied equally according to their degeneracy. Recalculate the hydrogen atom partition function and explain how the relation obtained in part c. is modified.

e. Suppose that the electrons and the protons interact according to the Debye shielded potential so that the charged particle free energy has an extra term $-\frac{1}{2}e^3 {N_e}^{5/2}\sqrt{4\pi/kT}$. How does this change the relation between the number of hydrogen atoms and the number of electrons at temperature T in the volume V?

4. The atoms in a diatomic molecule have masses m_1 and m_2 and are at a distance r from each other. The molecules are at temperature T.

a. Determine the rotational energy levels and the degeneracies.

b. Determine the rotational partition function. Obtain its value in the high temperature limit ($kT \gg \triangle E$, the separation between the rotational energy levels).

c. Determine the specific heats in the low temperature and the high temperature limits.

5. The one-dimensional Ising model is a chain of N spins, each spin interacting only with the two nearest neighbors and with and external field. We impose the periodic boundary condition by defining $s_{N+1} \equiv s_1$, thus making the topology of the chain that of a circle The energy for the configuration specified by $\{s_1, s_2, \ldots, s_N\}$ is

$$E = -\epsilon \sum_{k=1}^{N} s_k s_{k+1} - B \sum_{k=1}^{N} s_k \ ,$$

where the interaction energy ϵ between nearest-neighbor pairs of spins and the external magnetic field B are given positive constants, and each s_i independently assumes the values ± 1. We will consider the limit $N \to \infty$.

a. Obtain the partition function and show that it can be written in terms of matrices.

b. Calculate the Helmholz free energy per spin.

c. Calculate the magnetization per spin and show that for all $T > 0$, the one-dimensional Ising model never exhibits ferromagnetism. Give physical reasons for this behavior.

6. Consider a gas of bosons of volume V and temperature T, whose numbers are not conserved.

a. Show that in general the thermodynamic potential Ψ can be written as pV.

b. Suppose that the bosons have energies ϵ_j. Determine the form of the thermodynamic potential and explain why the chemical potential vanishes.

c. Assume that, for waves confined within the volume L^3, the dispersion relation is $\epsilon = \alpha k^2$. Determine the density of states for this system, and obtain the integral form for Ψ.

d. Determine the temperature dependency of the pressure and energy for this system.

Hint: $\Psi = U - TS - \mu N$.