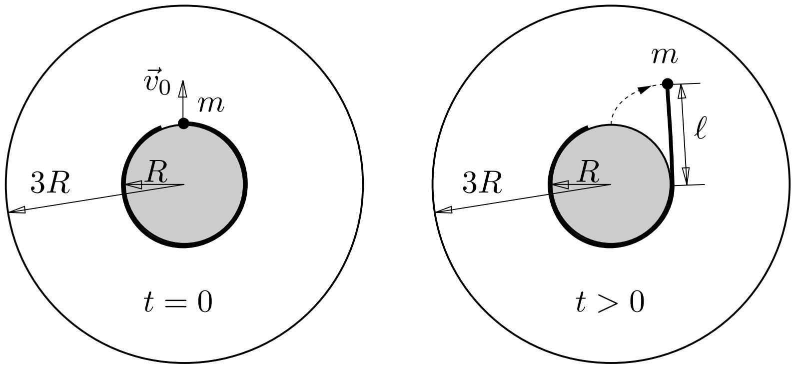
**1.** A bead of mass $m$, under no external force, is attached by a massless inextensible cord which is completely wound aroud a cylinder of radius $R$. This cylinder is placed within a concentric cylindical shell, of radius $3R$. A radially directed kick sends the bead spiraling outward with initial velocity $\vec{v}_0$, unwinding the cord as shown:




[7pts] a. Using the length of the unwound piece of the cord, $\ell$, as a generalized coordinate, write down the Lagrangian and determine the equation of motion.

[7pts] b. Find the trajectory $\ell = \ell(t)$ of the bead.

[7pts] c. Find the angular momentum of the bead about the axis of the cylinder and the kinetic energy after a time $t$.

[4pts] d. Find the time when the bead will hit the outer cylinder.

**2.** This is a relativistic problem with the speed of light, $c$, set equal to 1. Energy means total energy including rest mass energy. In the laboratory frame (LAB), a particle of mass $m_1$ and momentum $p_L$ (and energy $E_L$) collides with a particle of mass $m_2$ at rest.

[10pts] a. The center of mass (CM) frame is moving at constant speed $\beta$ relative to the LAB frame. We define CM as the frame where $m_1$ and $m_2$ have equal and opposite momentum. Show that

$$\beta = \frac{p_L}{m_2 + E_L} \ .$$

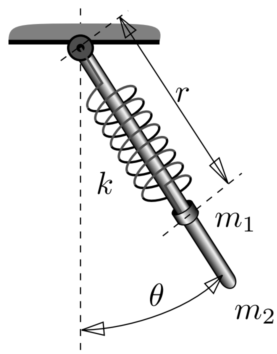
(Hint: Use Lorentz transformation between the LAB and the CM frames.)

[8pts] b. Show that the total energy, $W$, of both particles in CM frame is given by the relation $W^2 = {m_1}^2 + {m_2}^2 + 2m_2E_L$.

[7pts] c. Show that, in the CM frame, in the non-relativistic limit

$$\beta = \frac{p_L}{m_1 + m_2} \ , \qquad p = \left(\frac{m_2}{m_1 + m_2}\right) p_L \ , \qquad W = m_1 + m_2 + \left(\frac{m_2}{m_1 + m_2}\right) \frac{p_L^2}{2m_1} \ .$$

**3.** A ring with mass $m_1$ slides over a uniform rod which has a mass $m_2$ and length $\ell$. The rod is pivoted at one end and hangs vertically. The ring is secured to the pivot by a massless spring with the spring constant $k$ and unstretched length $r_0$, and is constrained to slide along the rod without friction. The rod and the ring are set into motion in a vertical plane. The position of the ring and the rod at time $t$ is given by $r(t)$ and $\theta(t)$, as shown in the figure.




[12pts] a. Write the Lagrangian for the system.

[5pts] b. Obtain the Hamiltonian.

[8pts] c. Obtain the differential eqution of motion.

**4.** A sphere of radius $R_1$ is constrained to roll without slipping on the lower half of the inner surface of a stationary hollow cylinder of radius $R_2$. The motion is confined to a plane perpendicular to the axis of the cylinder.

[4pts] a. Determine the moment of inertia of the sphere.

[2pts] b. What generalized coordinate may be used to describe the motion?

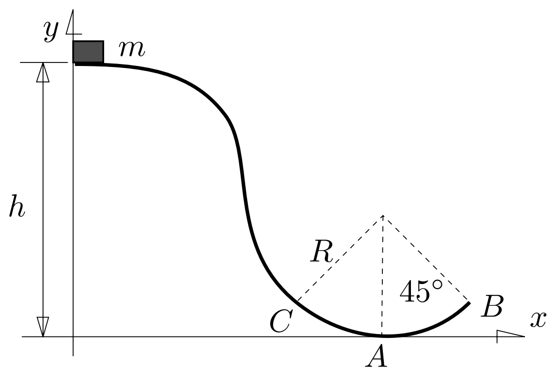
[6pts] c. Determine the Lagrangian function.

[3pts] d. Determine the equation of the constraint.

[5pts] e. Find the equations of motion.

[5pts] f. Find the Frequency of small oscillations.

**5.** A block of mass $m$ slides down a frictionless incline as in the figure below. The block is released at height $h$ above the bottom of the loop ($CAB$ is an arc of a circle of radius $R$).



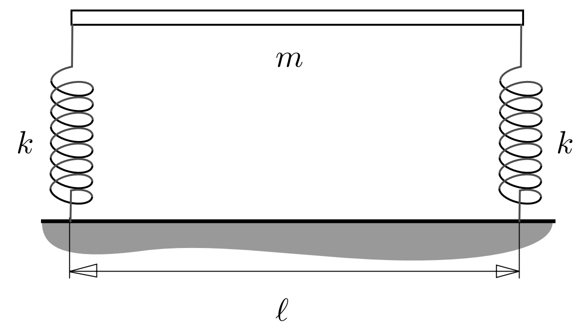

[7pts] a. Find the force of the inclined track on the block at the bottom (point A).

[7pts] b. Find the force of the track on the block at point $B$.

[4pts] c. How far away from point $A$ does the block land on level ground (the $x$ axis)?

[7pts] d. Sketch the potential energy $U(x)$ of the block (to the point $B$). Indicate the total energy on the sketch.

**6.** A uniform rigid rod of length $\ell$ and mass $m$ is supported at its ends by identical springs with spring constant $k$. The rod is set in motion by depressing one end of the rod by a small distance $a$ and then releasing it from rest. The motion is confined to the vertical plane containing the rod.




[4pts] a. Calculate the moment of inertia for the rod about an axis perpendicular to the bar and passing through the center of mass.

[7pts] b. Set up the equations of motion for the oscillating rod.

[7pts] c. Calculate the normal frequencies of oscillation of this system.

[7pts] d. Find the corresponding normal modes and sketch them.

**1.** A particle of charge $q$ and mass $m$ enters, at constant speed $v_0$, a cyclotron with a homogeneous magnetic field $\vec{B}$, perpendicular to the direction of motion of the particle.

[5pts] a. Determine the radius and frequency of the circular orbit which the particle will assume within the cyclotron.

[5pts] b. Determine the (non-relativistic) rate of radiative energy loss per unit time (the Larmor formula: up to the numerical factor $\frac{2}{3}$, it can be determined on dimensional and invariance grounds).

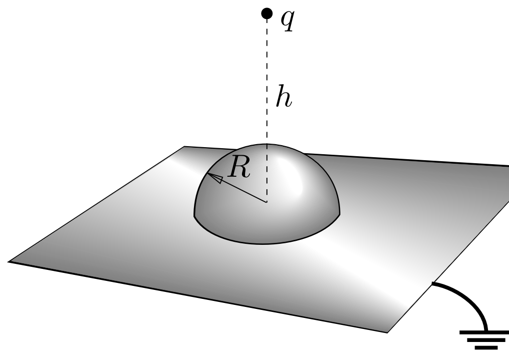
Generalize now the Larmor formula to the relativistic case as follows.

[5pts] c. By transforming from the instantaneous rest frame to the lab frame, prove that the rate of change of energy is a Lorentz invariant.

[5pts] d. As necessary, replace the factors in the Larmor formula with corresponding Lorentz invariants.

[5pts] e. Find the relativistic energy loss per (circular) revolution and express this in terms of the original variables and constants of nature.

**2.** A grounded conductor has the shape of an infinite horizontal plane, with a hemispherical bulge of radius $R$ (see the figure below). A point-charge $q$ is placed at a distance $h > R$ above the center of the hemisphere.




[12pts] a. Using the method of images, determine the total electrostatic potential.

[7pts] b. Determine the electrostatic force on the original charge.

[6pts] c. Determine the lowest non-zero term in the multipole expansion of the electrostatic potential.

[4pts] **3.** a. Write down the Maxwell's equations, in their integral form, for the fields $\vec{E}$ and $\vec{B}$.

[6pts] b. Use the appropriate Maxwell's equation to find the $\vec{B}$ field of a static current $I$ in an infinite, straight wire with round cross-section of radius $a$. Find $\vec{B}$ both inside and outside the wire. Express $\vec{B}$ in terms of the unit vector(s) of the coordinate system you choose to solve the problem.

[3pts] c. Make a sketch depicting the flux density variation with the (perpendicular) distance from the axis of the wire.

[6pts] d. Find the vector magnetic potential $\vec{A}$ in the plane bisecting a straight piece of a thin wire of finite length $2L$ in free space, carrying a steady current $I$.

[6pts] e. Find $\vec{B}$ from $\vec{A}$, using the results from part d. Show that the expression for $\vec{B}$ reduces (in a suitable limit) to the results obtained, in part b., for $\vec{B}$ outside the infinitely long wire.

**4.** Two long coaxial conducting cylindrical shells, with vertical axis of symmetry and of radii $a$ and $b$ are lowered vertically into a liquid dielectric. A potential difference $V$ is maintained between the two shells.

[5pts] a. Determine the electrostatic field between the two cylinders in vacuum, i.e., before the immersion into the dilelectric.

[5pts] b. Find the electrostatic energy $U_{\text{vac.}}$ of the system (still in vacuum).

[7pts] c. If the dielectric liquid rises a height $h$ within the space between the two shells, find the electrostatic energy $U_{\text{liq.}}$ in the region occupied by the liquid.

[8pts] d. From the excess electrostatic energy which lifts the liquid against gravity, show that the electric susceptibility of the liquid is

$$\chi_e = \frac{\epsilon - 1}{4\pi} = \frac{(b^2 - a^2)\,\rho\, g\, h\, \ln\left(\frac{b}{a}\right)}{V^2} ,$$

where $\rho$ is the density of the liquid and $g$ the gravitational acceleration. For air, $\epsilon = 1$, $\chi_e = 0$.

**5.** A plane electromagnetic wave of frequency $\omega$ and wavenumber $k$ propagates in the positive $z$ direction. For $z < 0$, the medium is air and the conductivity is $\sigma_a = 0$. For $z > 0$, the medium is a lossy dielectric, with dielectric constant $\kappa$ and $\sigma_d > 0$. Assume that both air and dielectric are nonmagnetic, and that $\vec{E}, \vec{B}$ are in the $(x, y)$-plane.

[10pts] a. Show that the dispersion relation (relation between $k$ and $\omega$) in the lossy medium is

$$k^2 = \frac{\omega^2}{c^2}\big(\kappa + i\frac{4\omega\sigma_d}{\omega}\big) \ .$$

[5pts] b. Find the values of $\eta$ and $\xi$ if $k$ is written as

$$k = \frac{\omega}{c}\big(\eta + i\xi\big) \ .$$

[5pts] c. Find the limiting value of $k$ for a very poor conductor ($\sigma_d \ll \kappa\omega$), and for a very good conductor ($\sigma_d \gg \kappa\omega$).

[5pts] d. Find the $e^{-1}$ penetration depth $\delta$ ("skin depth"), for the plane wave power in the case $\sigma_d \gg \kappa\omega$.

**6.** A resistor of length $L$, cross-section area $A$ and conductivity $\sigma$ is placed in an electric field which causes an electric current to flow uniformly along the length of the resistor.

[2pts] a. Calculate the resistance of the resistor.

[7pts] b. A small defect is introduced in the middle of the body of the resistor, having a small length $b$, a small uniform cross-section $a$ and conductivity $\sigma_d$. Calculate the effective total resistance.

Approximate now the defect by a small ball of radius $b$ within an infinitely big resistor ($L, A \to \infty$), and assume the current density to remain uniform far away from the defect.

[7pts] c. List carefully all the boundary conditions on the electric field.

[9pts] d. Apply the boundary conditions to calculate the electric field and current within the defect.

Hint: You may use the azimuthal symmetry to write down a general expression for the potential in the form of a series, and then use the boundary conditions to determine the coefficients.

**1.** An operator $Q$ satisfies the relations

$$\big[\,[\,Q\,,\,\vec{J}^2\,]\,,\,\vec{J}^2\,\big] = \tfrac{1}{2}\big(Q\vec{J}^2 + \vec{J}^2 Q\big) + \tfrac{3}{16}Q\ , \qquad [\,Q\,,\,J_z\,] = m_q Q\ ,$$

where $\vec{J}$ is the usual (total) angular momentum (vector) operator and $J_z$ the component in the $z$ direction.

[6pts] a. For the matrix element $\langle j',m'|\,Q\,|j,m\rangle$ to be non-zero, use the first relation to determine the allowed values $\triangle j = j' - j$.

[6pts] b. For the matrix element $\langle j',m'|\,Q\,|j,m\rangle$ to be non-zero, use the second relation to determine the allowed values $\triangle m = m' - m$ in terms of $m_q$.

[3pts] c. Given your results for a. and b., what are the two possible values for $m_q$?

[5pts] d. Calculate $\langle j',m'|\,[\,Q\,,\,\vec{J}^2\,]\,|j,m\rangle$ in terms of $\langle j',m'|\,Q\,|j,m\rangle$, $j$ and $\triangle j$.

[5pts] e. Writing $Q$ and $\bar{Q}$ for the two operators corresponding to the two possible values of $m_q$, prove that $Q\bar{Q}$ and $\bar{Q}Q$ commute with $J_z$.

Hint: "Sandwich" the given relations between $\langle j',m'|$ and $|j,m\rangle$.

[6pts] **2.** a. Calculate the energy levels for a particle of mass $m$ in a one dimensional square well of width $2a$ and of infinite depth:

$$V(x) = \begin{cases} +\infty & \text{for } |x| > a, \\ 0 & \text{for } |x| < a. \end{cases}$$

[7pts] b. Find expressions for the normalized eigenfunctions for the energy levels calculated in part a. Sketch the eigenfunction for the second excited state.

[5pts] c. Suppose we now make a stepped potential well given by

$$V(x) = \begin{cases} +\infty & \text{for } |x| > a, \\ 0 & \text{for } \frac{a}{3} < |x| < a, \\ +\delta & \text{for } |x| < \frac{a}{3}. \end{cases}$$

Use first order perturbation theory to find out the lowest energy level for this potential, assuming that $\delta$ is small compared to the energy of the lowest level.

[5pts] d. Find the wave function (up to first order perturbation theory) for the lowest energy level calculated in part c.

[2pts] e. Compare this perturbed wave function with the corresponding unperturbed wave function for infinite square well by means of a sketch.

**3.** Pion-nucleon scattering at low energies can be qualitatively described by an effective interaction potential of the form

$$V = \frac{g^2}{4\pi}\frac{e^{-r/\rho}}{r}\vec{I}_\pi \cdot \vec{I}_N \ ,$$

where $g$ and $\rho$ are the interaction constant and the effective range (constant), and $r$ is the pion-nucleon distance. Defining a total isospin, $\vec{I} \stackrel{\text{def}}{=} \vec{I}_\pi + \vec{I}_N$, scattering processes can be classified by $I(I+1)$, the eigenvalues of $\vec{I}^2$.

[5pts] a. For the differential cross-section, in the usual approximation $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$, specify the scattering amplitude in the Born approximation, for the above potential. (Up to the numerical factor $\frac{1}{2\pi}$, the constant coefficients can be determined by dimensional and general arguments.)

[5pts] b. Calculate the ratio of (total) cross-sections $\sigma_{\frac{3}{2}} : \sigma_{\frac{1}{2}}$, where $\sigma_I$ is the cross-section for a scattering in a state of total isospin $I$.

[5pts] c. Calculate the spatial (isospin-independent) factors in the scattering amplitude for the above potential.

[5pts] d. Calculate the isospin factor in the scattering amplitude for the scattering processes $\pi^+{+}p \to \pi^+{+}p$, $\pi^-{+}p \to \pi^-{+}p$ and $\pi^-{+}p \to \pi^0{+}n$.

[5pts] e. Calculate the total cross-sections for the three scattering processes in d.

Know: $|\pi^+\, p\rangle = |\frac{3}{2},\frac{3}{2}\rangle$, $|\pi^-\, p\rangle = \sqrt{\frac{1}{3}}\,|\frac{3}{2},-\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}\,|\frac{1}{2},-\frac{1}{2}\rangle$, $|\pi^0\, n\rangle = \sqrt{\frac{2}{3}}\,|\frac{3}{2},-\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}\,|\frac{1}{2},-\frac{1}{2}\rangle$;
also: $|\pi^\pm\rangle = |1,\pm1\rangle$, $|\pi^0\rangle = |1,0\rangle$, $|p\rangle = |\frac{1}{2},\frac{1}{2}\rangle$ and $|n\rangle = |\frac{1}{2},-\frac{1}{2}\rangle$.

**4.** Consider a general 1-dimensional problem: a non-relativistic quantum particle moving along $x$, in an arbitrary potential $V(x)$.

[3pts] a. Write down the Hamiltonian and state the canonical commutation relation, $[x,p] =?$

[4pts] b. Evaluate $[H,x]$ and $[x,[H,x]]$.

[3pts] c. Calculate the expectation value of $\big[x,[H,x]\big]$ in the ground state, $\psi_0$.

[5pts] d. By expanding the double commutator and inserting complete sets of intermediate states, obtain the Thomas-Reiche-Kuhn sum rule:

$$\sum_{n=0}^{\infty} \frac{2m}{\hbar^2}(E_n - E_0)\big|\langle n|x|0\rangle\big|^2 = 1\ .$$

[5pts] e. By expanding the triple commutator $[x,[x,[H,x]]]$ and inserting complete sets of intermediate states prove:

$$\sum_{k,n=0}^{\infty} (E_k - E_n)\langle 0|x|k\rangle\langle k|x|n\rangle\langle n|x|0\rangle = 0\ .$$

[5pts] f. For a 3-dimensional system with coordinates $x_1, x_2, x_3$, derive the generalization:

$$\sum_{k,l,n=0}^{\infty} \frac{2m}{\hbar^2}(E_{k,l,n} - E_{0,0,0})\,\Re e\left\{\langle 0,0,0|x_i|k,l,n\rangle\langle k,l,n|x_j|0,0,0\rangle\right\} = \delta_{ij}\ ,$$

where $\Re e(z)$ denotes the real part of $z$.

(Ehrenfest's theorem relates the first few results to the Hamilton-Heisenberg equations of motion.)

**5.** Consider a nonrelativistic quantum particle moving in a 1-dimensional box with impenetrable walls placed at $x = 0$ and $x = L$.

[7pts] a. Write down the Schrödinger equation, find the complete set of normalized stationary states and the corresponding energy eigenvalues.

[5pts] b. Assume now that the wall at $x = L$ is very slowly (so that the quantum state of the particle adapts continuously) moved out to $x = 2L$. Determine the new enerly levels.

[7pts] c. Assume now that the particle is in the ground state when the wall at $x = L$ is instantaneously moved to $x = 2L$. Calculate the probability that the particle is in the ground state of the stretched system.

[6pts] d. Finally, consider the original particle in the original box, but with a perturbation $H' = Axe^{-(t/\tau)^2}$. If the particle was in the $n^{th}$ (unperturbed) state at time $t = -\infty$, find the probability that it will be in another, $k^{th}$ state at $t = +\infty$.

You may find the integrals $\int_0^\pi \mathrm{d}\phi \sin(k\phi)\, \phi\, \sin(n\phi) = (\frac{\pi}{2})^2 \delta_{k,n} - [1-\delta_{k,n}] \frac{2kn}{(k^2-n^2)^2} [1 - (-1)^{k+n}]$, $\int_0^\pi \mathrm{d}\phi \sin(k\phi) \sin(n\phi) = \frac{\pi}{2}\delta_{k,n}$ and $\int_{-\infty}^{+\infty} \mathrm{d}z e^{-z^2} = \sqrt{\pi}$ useful.

**6.** A particle has the wave-function:

$$\psi(r,\theta,\phi) = \sqrt{\frac{5}{16\pi}} f(r) \sin^2\theta \left(1+\sqrt{14}\cos\theta\right) \cos 2\phi ,$$

where $f(r)$ is a normalized radial function.

[10pts] a. Express the angular part of this wave-function as a superposition of the spherical harmonics $Y_\ell^m(\theta,\phi)$.

[5pts] b. Calculate the probabilities that a measurement of $\vec{L}^2$ and $L_z$ will yield the values in the table:

| **Operator** | **Measurements and Probabilities** | | | | | | |
|---|---|---|---|---|---|---|---|
| $\vec{L}^2$ **:** | 2 | 6 | 12 | | | | |
| **Prob. =** | | | | | | | |
| $L_z$ **:** | $-3$ | $-2$ | $-1$ | 0 | $+1$ | $+2$ | $+3$ |
| **Prob. =** | | | | | | | |

[5pts] c. Calculate the expectation values $\langle\psi|\vec{L}^2|\psi\rangle$ and $\langle\psi|L_z|\psi\rangle$.

[5pts] d. Calculate the rms uncertainties in $\vec{L}^2$ and $L_z$.

Some spherical harmonics:

$$Y_0^0 = \sqrt{\tfrac{1}{4\pi}} , \quad Y_1^0 = \sqrt{\tfrac{3}{4\pi}}\cos\theta , \quad Y_2^0 = \sqrt{\tfrac{5}{16\pi}}(3\cos^2\theta - 1) , \quad Y_3^0 = \sqrt{\tfrac{7}{16\pi}}(5\cos^3\theta - 3\cos\theta) ,$$

$$Y_1^1 = -\sqrt{\tfrac{3}{8\pi}}\sin\theta\, e^{i\phi} , \quad Y_2^1 = -\sqrt{\tfrac{15}{8\pi}}\sin\theta\,\cos\theta\, e^{i\phi} , \quad Y_3^1 = -\sqrt{\tfrac{21}{64\pi}}(5\cos^2\theta - 1)\sin\theta\, e^{i\phi} ,$$

$$Y_2^2 = \sqrt{\tfrac{15}{32\pi}}\sin^2\theta\, e^{2i\phi} , \quad Y_3^2 = \sqrt{\tfrac{105}{32\pi}}\sin^2\theta\,\cos\theta\, e^{2i\phi} , \quad Y_3^3 = -\sqrt{\tfrac{35}{64\pi}}\sin^3\theta\, e^{3i\phi} .$$

**1.** Consider a system of $N \gg 1$ non-interacting particles in which the energy of each particle can assume two and only two disticnt values: 0 and $E$ ($E > 0$). Denote by $n_0$ and $n_1$ the occupation numbers of the energy levels o and $E$, respectively. Let the fixed total energy of the system be $U$.

[3pts] a. Write down the total number of available states $\Omega$.

[6pts] b. Use Stirling's approximation to obtain the entropy of the system $S = S(N, n_0, n_1)$.

[2pts] c. Write down expressions for the total number $N$ of particles and total energy $U$ of the system.

[7pts] d. For a constant number of particles, determine the temperature $T = T(U, E, N)$.

[7pts] e. For what range of values of $n_0$ is $T < 0$?

**2.** Consider a heteronuclear diatomic molecule with moment of inertia $I$ and undergoing only rotational motion.

[2pts] a. Remembering that the rotation about the axis joining the two atoms is irrelevant, determine the total number of rotational degrees of freedom that the molecule possesses.

[3pts] b. Use the equipartition theorem to find the classical average internal energy $\langle E \rangle$. What is the specific heat $C(T)$ of the molecule at temperature $T$ according to classical statistical mechanics?

[4pts] c. In quantum mechanics, the molecule has energy levels $E_j = \frac{\hbar^2}{2I} j(j+1)$, $j = 0, 1, 2, \ldots$ Each $j$-level is $(2j+1)$-fold degenerate. Use quantum statistical mechanics to write down expressions for the partition function $Z$ and the average energy $\langle E \rangle$ of the system as a function of temperature $T$.

[8pts] d. By simplifying the (defining) expressions in c., show that for high temperatures

$$\langle E \rangle \approx 3E_1 \, e^{-\frac{E_1}{kT}} \ , \qquad \text{and} \qquad C(T) \approx \frac{3{E_1}^2}{kT^2} \, e^{-\frac{E_1}{kT}} \ .$$

In what range of temperatures is this result valid?

[8pts] e. By simplifying the (defining) expressions in c., show that for low temperatures

$$\langle E \rangle \approx kT \ , \qquad \text{and} \qquad C(T) \approx k \ .$$

In what range of temperatures is this result valid?

**3.** A partition initially divides a chamber of total volume $V_0$ which is otherwise surrounded by insulating walls. On one side of the partition are $N$ molecules of a monoatomic ideal gas of atomic number $A$ and at absolute temperature $T$ and volume $V$. The other side of the chamber is in vacuum. The partition is then ruptured, permitting the gas to expand freely throughout the chamber.

[7pts] a. Find the final temperature $T_f$, pressure $P_f$, and entropy change $\triangle S$ of the gas.

[4pts] b. What is the probability that all the molecules of the gas be found by chance within the initial volume $V$ at some time long after the partition is ruptured?

[7pts] c. Suppose instead that the gas expands from the initial volume $V$ to the final volume $V_0$ by pushing slowly and reversibly on the partition (which now acts as a piston) rather than through free expansion into vacuum. What would now be the final temperature $T_f$, final pressure $P_f$, and the entropy change $\triangle S$ of the gas?

[7pts] d. Show that the work done is given by

$$W = \frac{PV - P_f V_0}{\gamma - 1} \ ,$$

where $\gamma$ is the ratio of the heat capacities of the gas at constant pressure and at constant volume.

**4.** Consider a non-relativistic quantum gas composed of completely ionized helium. To a first approximation, assume that this is a white dwarf star such as Sirius B. The number, $N$, of helium atoms can be estimated by using its estimated mass, $M = 2.09\times10^{30}$ kg, and radius $R = 5.57\times10^{3}$ km. Since there are 4 nucleons per helium atom, neglecting the mass of the electrons, we get $N = 1.25\times10^{57}$ atoms ($m_p \approx m_n \approx m = 1.67\times10^{-27}$ kg).

[3pts] a. Starting with the Fermi-Dirac distribution function, obtain an expression of the Fermi energy at $T{=}0$ for an ideal gas of electrons and evaluate it for the conditions of Sirius B.

[2pts] b. Calculate the Fermi temperature for Sirius B. Is this larger than observed temperature of $2\times10^{7}$ K (implying that the electron gas is degenerate)?

[5pts] c. Obtain an expression for the internal energy, $U_e$, of the electron gas at $T{=}0$ K. The total internal energy is $U_e + U_g$ where $U_g$ is the gravitational internal energy of the He nuclei, $U_g = -\frac{3}{5}N^2(Gm^2/R)$.

[5pts] d. Find the radius which minimizes the total internal energy of the star. Calculate the radius of Sirius B and compare with the observed radius $R = 5.57\times10^{3}$ km.

[5pts] e. As the helium is burned, the star begins to collapse, the electron density rises and the Fermi energy increases until it exceeds the electron mass. At this point you must use relativistic mechanics to describe electrons. Assuming extreme relativistic conditions, the energy and momentum are connected by the speed of light, $E = cp$. Recalculate the Fermi energy for this case. Use the fact that the mean energy of the electron gas is, approximately, the mean momentum times $c$, to obtain a new equation for the free energy of the star.

[5pts] f. From part d., calculate the number of helium nuclei at which the gravitational energy equals the relativistic electron energy. This is the critical number. Show that the critical mass of a star is $3.4\times10^{30}$ kg.

(The electron mass is $m_e = 9.108\times10^{-31}$ kg; $\hbar = 1.055\times10^{-34}$ Js; $G = 6.67\times10^{-11}$ Nm$^2$/kg$^2$.)

**5.** Consider an ideal monoatomic gas. The system is a volume $V$ of gas surrounded by a much larger volume of gas which serves as a reservoir with constant temperature $T$ and chemical potential $\mu$. The partition function for a single gas atom is given by

$$Z = V\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}} .$$

[7pts] a. Calculate the grand partition function and the grand potential for the system.

[7pts] b. Find the average number of particles in the volume $V$, the entropy, and the pressure of the gas.

[4pts] c. Obtain the equation of state of the system.

[7pts] d. Suppose your system consists of proton-electron plasma. Assume this system may be found in three possible states. In the first state the hydrogen is ionized ($H^+$) and has energy chosen as zero. In the second state, the atom contains one electron ($H$) and has an energy $\varepsilon_1$. In the third state, the atom contains two electrons ($H^-$) and has an energy $\varepsilon_2$. The atom may exchange electrons with the surrounding atoms, but the hydrogen as a whole is neutral.

From the grand partition function of this system, show that $\mu = \frac{1}{2}\varepsilon_2$ so that the gas is neutral, that is, $\langle N \rangle = 1$.

**6.** A solid contains $N$ mutually noninteracting nuclei of spin 1. Each nucleus can therefore be in any of three quantum states labeled by the quantum number $m$, where $m = 0, \pm 1$. Because of electric interactions with internal fields in the solid, a nucleus in state with $m = 1$ or in the state with $m = -1$ has the same energy $E > 0$, while its energy in the state with $m = 0$ is zero.

[6pts] a. Write down the partition function $Z$, and find the Helmholz free energy $F$ for this system.

[6pts] b. Obtain espressions for the entropy $S$ of the $N$ nuclei as a function of temperature $T$.

[6pts] c. Obtain and expression for the energy $U$, as a function of $T$.

[7pts] d. Derive an expression for the heat capacity $C$ in the limit $E \ll kT$.