The Howard University
Department of Physics and Astronomy

Master of Science Comprehensive and Doctor of Philosophy Qualifying Exam

Part 1: Classical Physics

August 18, 1997

Of the included ten problems, circle *eight* numbers below to indicate which of your attempted solutions should be graded; they *must* include at least two of 1, 2, 3, at least two of 4, 5, 6, and at least two of 7, 8, 9, 10 :

$$ \boxed{1\ 2\ 3} \quad \boxed{4\ 5\ 6} \quad \boxed{7\ 8\ 9\ 10} $$

1. Write in you code-letter here: ☐ , and staple this cover sheet on the top of your answer sheets. (Put the code-letter and a page number also in the top-right corner of each submitted answer sheet.
2. Write only on one side of the answer sheets.
**3.** Start each problem with a new answer sheet.

Good Luck!

1. A mass m_1, with an initial velocity v_0, strikes a mass-spring system m_2, initially at rest but able to recoil. The spring is massless, with spring constant k. There is no friction.
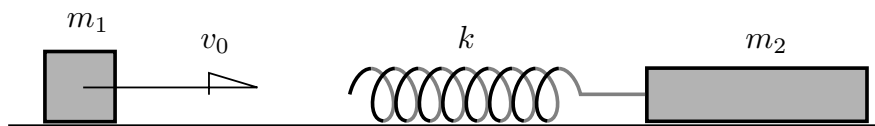
[12pts]   a. What is the maximum compression (A) of the spring?

[13pts]   b. If, long after the collision, both objects travel in the same direction, what are the final velocities v_1 and v_2 of m_1 and m_2, respectively?

2. A small particle of mass m slides, without friction, on the inside of a hemispherical bowl, of radius R, that has its axis parallel to the gravitational field g. Use the polar angle and the azimuthal angle to describe the location of the particle (which is to be treated as a point particle).
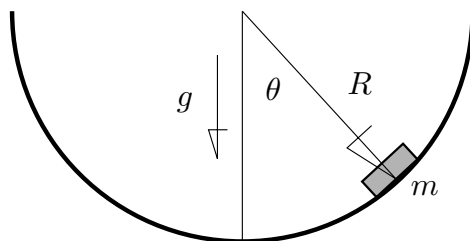
[7pts]   a. Write the Lagrangian and the Hamiltonian for the motion.

[6pts]   b. Obtain the generalized momenta p_θ and p_ϕ.

[6pts]   c. Determine the Hamilton's equations of motion.

[6pts]   d. Combine the equations in (c) so as to produce one second-order differential equation for θ as a function of time. If $\theta = \theta_0$ and $\dot{\theta} = 0$ at all times, calculate the velocity (magnitude and direction).

3. A particle of mass m moves under the action of a force whose potential is given by $V(r) = Kr^4$ where $K > 0$.

[7pts]   a. Calculate the force $\vec{F}(r)$ and make plots of both $|\vec{F}(r)|$ and $V(r)$.

[7pts]   b. Make a plot of the effective potential and discuss the motion of the particle without solving the equations of motion, for the cases $E < 0$, $E = 0$, and $E > 0$.

[5pts]   c. Find the values of total energy E, the Lagrangian function L and the radius of a circular orbit.

[3pts]   d. Calculate the period of this circular motion.

[3pts]   e. Calculate the period of small radial oscillations, that is, the period of the motion when the particle is slightly disturbed from the circular orbit.

4. Two infinite radial planes have an interior angle α. An infinitesimal insulating gap exists at their juncture, $\rho = 0$. The plate at $\phi = 0$ is held at the potential $V = 0$ and the one at $\phi = \alpha$ at the potential $V = V_0$.

[10pts]   a. Find the potential field in the region between the two plates.

[10pts]   b. Find the electric field $\vec{E}$ in the region between the plates.
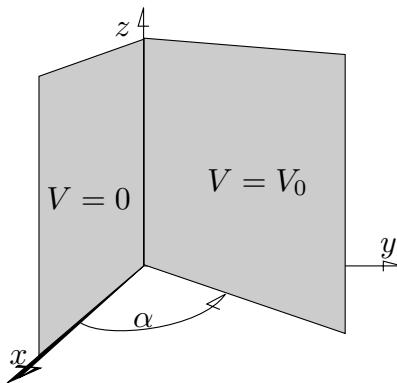
Based on the answers above,

[3pts]   c. The magnitude $|\vec{E}|$ is a function of (circle all that apply)

$$(1)\ \rho\ , \qquad (2)\ \phi\ , \qquad (3)\ z\ .$$

[2pts]   d. The vector $\vec{E}$ is a function of (circle all that apply)

$$(1)\ \rho\ , \qquad (2)\ \phi\ , \qquad (3)\ z\ .$$

(Careful here. Think about unit vectors!!)

5. An alternating current $I = I_0 cos(\omega t)$ flows down a long straight wire and back along a coaxial conducting cylinder of radius R.

[12pts]   a. In what direction does the induced electric field point (radial, circumferential, or longitudinal)?

[13pts]   b. Find $\vec{E}$ as a function of r (the normal distance from the axis).
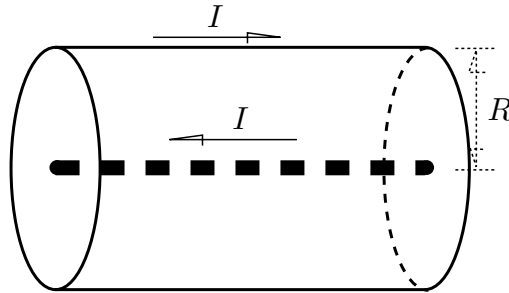
6. Given are the Maxwell equations in Gaussian units:

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \ , \qquad \vec{\nabla} \cdot \vec{D} = 0 \ ,$$

$$\vec{\nabla} \cdot \vec{B} = 0 \ , \qquad \vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t} \ ;$$

a vector identity $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$, and that $c = 3 \times 10^{10}$ cm/s.

It is desired to calculate the reflectivity, R, from a metal surface at normal incidence. Same values of ε and μ may be used for vacuum and metal. However, there is a large conductivity σ associated with the metal.

[10pts]   a. What are the wave equations for a plane wave of frequency ω in the vacuum and the metal? What are the solutions of these equations?

[10pts]   b. How do you match the boundary conditions at the metal surface?

[5pts]   c. How do you calculate R? For visible light, $\omega = 3 \times 10^{15}\,\mathrm{s}^{-1}$. Estimate R for copper for which $\sigma = 5 \times 10^{17} \mathrm{statohm}^{-1}\mathrm{cm}^{-1} = 6 \times 10^{7} \mathrm{ohm}^{-1}\mathrm{m}^{-1}$.

7. One mole of a bi-atomic ideal gas (such as O_2) undergoes the following cycle: Starting at pressure P_a and volume V_a, it first expands isothermally to P_b and V_b. In the second stage the gas contracts at constant pressure to V_a. In the third stage of the cycle it returns to its initial state P_a, V_a. (All these transformations are reversible.)

[7pts]   a. Draw the PV diagram of the cycle.

Using P_a, V_a, P_b and V_b, find:

[7pts]   b. the work done by the gas in each of the three stages.

[7pts]   c. the change in the internal energy of the gas in each of the three stages.

[4pts]   d. the net heat exchange with the surroundings in the entire cycle.

8. Molecules of air are moving around randomly all the time. However, fortunately, we have never encountered a situation in which all the air molecules moved to the other half of the room, while our half remained without air. As a matter of fact, the amount of air is the same in both halves of the room. The following model is constructed to explain the situation.

There are N imaginary cells and n molecules of air in the room, and $N \gg n \gg 1$. The identical molecules are moving freely among the cells, and at any given time any cell may contain one molecule or it may be empty. A micro-state is defined as any arrangement of molecules in the room. Macro-states are characterized by K, the number of molecules in the left half of the room ($0 \leq K \leq n$).

[10pts]   a. How many micro-states belong to the macro-state K?

[10pts]   b. Calculate which macro-state has the largest number of micro-states.

[5pts]   c. How do these results explain the fact that there are equal amounts of air in the two halves of the room?

You may want to use:

$$\ln(x!) \approx x \ln(x) - x , \qquad \text{for } x \gg 1 , \qquad \binom{p}{q} = \frac{p!}{q!(p-q)!} = \prod_{k=1}^{q} \left(\frac{p-k+1}{k} \right) .$$

9. A sheet of ice has frozen over a lake. The upper surface of ice is at $-20°$C. The thermal conductivity of ice is $\kappa = 5 \times 10^{-3}$ cal/cm sec $°$C. The latent heat of fusion of water is $L = 80$ cal/g and the density of water is $\rho = 1.0\,$g/cm^3.

[13pts]   a. At what rate the thickness of ice increasing?

[12pts]   b. How long will it take to double the thickness?

10. The tension H in a rubber band stretched at constant length, L, is related to the temperature, T, by $H = A(L)T$, where $A(L) > 0$ is a function of L only. Also, $\left(\frac{\partial A}{\partial L}\right)_T > 0$.

[7pts]   a. Show that the internal energy is a function of T only.

[7pts]   b. Show that the entropy decreases as the band is stretched isothermally.

[7pts]   c. Show that T increases for adiabatic stretching.

[4pts]   d. Show that L decreases for heating at constant tension.

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Part 2: Modern Physics

August 20, 1997

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Good Luck!

1. Consider a relativistic particle of rest mass m, momentum p, and total energy E.

[8pts]   a. Show that $E^2 = p^2 + m^2 c^4$.

[8pts]   b. Show that the "length" of the energy-momentum four-vector of a particle is Lorentz-invariant.

[9pts]   c. A proton-proton collision can create a π^0 meson as an additional particle in the final state, if there is sufficient energy. The observed reaction is $p + p \rightarrow p + p + \pi^0$. If the initial state consists of a proton of kinetic energy K colliding with a proton at rest, calculate the minimum value of K for which the reaction may occur.

2. The atomic number of the sodium (Na) atom is $Z = 11$.

[8pts]   a. Write down the electronic configuration for the ground state of the Na atom showing in standard (s,p,d.....) notation the assignment of all the electrons to various one-electron states. Give the standard spectroscopic notation for the ground state of the Na atom in the form $n^{2S+1}L_J$.

[8pts]   c. The lowest frequency line in the absorption spectrum of the Na atom gives rise to a doublet. These two lines are called the D lines of sodium and occur at wavelengths of 588.9950 and 588.5924 nm, respectively. What is the mechanism responsible for the two D lines in the sodium optical spectrum? Draw an energy-level diagram showing the radiative transitions responsible for these lines.

[9pts]   c. In the sodium atom the single valence electron sees the core as a spherically symmetric distribution of charge. Every sodium energy-level for nonzero angular momentum has a fine structure associated with it due to spin-orbit interaction of the valence electron. Using central-field model, write down the spin-orbit Hamiltonian, assuming a Coulomb potential $U(r) = -\frac{Ze^2}{r}$. Determine explicitly the spin-orbit perturbation energy and show that this is proportional to n^{-3}, where n is the principal quantum number.

3. Consider the magnetic moment of a one-electron atom.

[7pts]   a. Show that the component of the magnetic moment, in L-S coupling, along the direction of the total angular momentum is given by $\mu_J = -\mu_b\, g\, \sqrt{j(j+1)}$, where

$$g = 1 + \frac{j(j+1) - \ell(\ell+1) + s(s+1)}{2j(j+1)} \ .$$

The quantum numbers ℓ, s and j are for the orbital, spin and total angular momenta, respectively, and μ_b is the Bohr magneton.

[7pts]   b. Atoms in a weak magnetic field B emit Zeeman spectrum. Show that the orientational potential energy is given by $\delta E = \mu_b B g m_j$.

[7pts]   c. Draw an energy level diagram for a one-electron atom in a weak magnetic field B showing the splitting of an excited $P_{3/2}$ state and the ground $S_{1/2}$ state.

[4pts]   d. Indicate the allowed transitions in the energy level diagram.

4. A Compton collision occurs in which the photon is scattered directly backwards. The scattered photon has half the energy of the incident photon.

[7pts]   a. Starting with the conservation laws, find the wavelength of the incident photon.

[7pts]   b. Find the energy of the recoil electron.

[7pts]   c. Find v/c for the recoil electron.

[4pts]   d. Show that the photon cannot lose all its energy in a collision with a free electron.

5. Consider an $L \times L \times L$ cube of metal, wherein the electrons may be treated as if comprised of an ideal gas confined in the cube.

[5pts]   a. Write down the wave-function for the electron states and the expression for the energy levels, $E_{\vec{n}}$, where $\vec{n} = (n_x, n_y, n_z)$.

[5pts]   b. Let $n = |\vec{n}|$. Determine the number of state between n and $n + dn$. Using the relation between $E_{\vec{n}}$ and $\vec{n}$, and $dE_{\vec{n}}$ and dn, eliminate n and dn and obtain the number of states, dN, within $[E, E + dE]$.

[5pts]   c. Determine the Fermi energy, that is, the energy of the highest occupied state.

[5pts]   d. Determine the average kinetic energy of these electrons.

[5pts]   e. Determine the pressure of this ideal gas of electrons.

6. Consider a particle trapped in a one-dimensional finite potential well: $V(x) = 0$ for $|x| > a$, and $V(x) = -V_0$ for $|x| < a$, where V_0 and a are both positive.

[6pts]   a. Sketch the potential, and the wave functions of the ground and first excited state.

[6pts]   b. Write down the Schrödinger equation and specify all the boundary-matching conditions on the stationary state wave functions.

[7pts]   c. Consider taking the "infinite potential well" limit by sending $V_0 \to \infty$. Explain why $H_1 = \frac{1}{2m}p^2 + V(x)$ is a bad choice for the Hamiltonian while $H_2 = H_1 + V_0$ is good. (This exemplifies renormalization.)

[6pts]   d. Find the energy levels and the wave functions of the stationary states in the infinite potential well with $H = \frac{1}{2m}p^2 + V(x)$ where $V(x) = 0$ for $|x| < a$, and $V(x) = -V_0$ for $|x| > a$.

7. Consider a particle trapped in a one-dimensional infinite potential well: $V(x) = 0$ for $|x| < a$, and $V(x) = \infty$ for $|x| > a$, where $a > 0$.

[5pts]   a. Sketch the potential, and the wave functions of the ground and first excited state.

[6pts]   b. Write down the Schrödinger equation and specify all the boundary-matching conditions on the stationary state wave functions.

[8pts]   c. Change the potential by introducing a δ-function term: $V(x) \to V(x) + \lambda\delta(x)$, with $\lambda > 0$. Determine the boundary-matching condition for the stationary functions at $x = 0$. Sketch the effect of this on the ground state wave-function.

[6pts]   d. Describe the effect of the infinitely opaque δ-function barrier ($\lambda \to \infty$ limit): find the stationary states and specify their energy spectrum.

8. A particle is moving freely within a "box" of length L, but is strictly confined to that box.

[6pts]   a. Two possible eigenfunctions for this particle are shown below. When the particle is in the state corresponding to the eigenfunction ψ_a, its total energy is $4\,\mathrm{eV}$. Describe the quantum states (principal quantum number) for the two eigenfunctions shown.
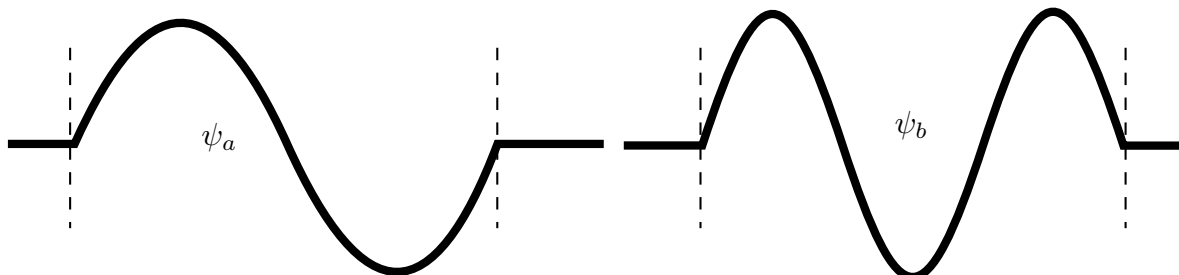
[7pts]   b. What is the total energy in the state corresponding to ψ_b?

[7pts]   c. Show that the fractional difference in the energy between adjacent eigenvalues is given by

$$\frac{E_{n+1} - E_n}{E_n} = \frac{2n + 1}{n^2} \ .$$

[5pts]   d. Starting with Schrödinger time independent equation, derive the form of the eigenfunction for the particle trapped in a box as described in part (a) above.

9. An electron is described by the wavefunction

$$\psi(x) = \begin{cases} 0 & \text{for } x < 0, \\ Ce^{-x/x_0}(1 - e^{-x/x_0}) & \text{for } x > 0, \end{cases}$$

where $x_0 = 1\,\mathrm{nm}$ and C is a constant.

[8pts]   a. Find the value of C that normalizes $\psi(x)$.

[8pts]   b. Where is the electron most likely to be found, that is, for what value of x is the probability of finding the electron the largest?

[9pts]   c. Calculate $\langle x \rangle$ for this electron and compare your results with the most likely position. Comment on any differences.

10. Consider a 2-dimensional harmonic oscillator with the Hamiltonian

$$H = \sum_{i=1}^{2} H_i \ , \qquad H_i = \tfrac{1}{2m}p_i{}^2 + \tfrac{1}{2}m\omega^2 x_i{}^2 \ .$$

[2pts]   a. State the canonical commutation relations $[x_i, p_j] =?$, and determine the commutation relations $[a_i, a_j]$ and $[a_i, a_j^\dagger]$ where

$$a_j \stackrel{\text{def}}{=} \sqrt{\frac{m\omega}{2\hbar}}\Big[x_j + \frac{i}{m\omega}p_j\Big] \ , \qquad a_j \stackrel{\text{def}}{=} \sqrt{\frac{m\omega}{2\hbar}}\Big[x_j - \frac{i}{m\omega}p_j\Big] \ .$$

[4pts]   b. Rewrite H in terms of a_i and $a_j^\dagger$.

[4pts]   c. Given a complete set of orthonomalized states $|n_1, n_2\rangle$ such that

$$a_1^\dagger\,|n_1, n_2\rangle = \sqrt{n_1{+}1}\,|n_1{+}1, n_2\rangle \ , \qquad a_1\,|n_1, n_2\rangle = \sqrt{n_1}\,|n_1{-}1, n_2\rangle \ ,$$

and similarly for $a_2^\dagger, a_2$, prove that $H\,|n_1, n_2\rangle = \hbar\omega(n_1{+}n_2{+}1)\,|n_1, n_2\rangle$.

[6pts]   d. Prove that $[a_i^\dagger a_j, H] = 0$ for all i, j by explicit calculation.

[6pts]   e. Prove that $[a_i^\dagger a_j, H] = 0$ for all i, j without calculating any commutators, but by carefully examining the action of each of the four operators $a_i^\dagger a_j$ on $|n_1, n_2\rangle$, and knowing that these states form a complete set.

[3pts]   f. Give a physical interpretation to the operators

$$L_+ \stackrel{\text{def}}{=} (a_1^\dagger a_2 + a_2^\dagger a_1) \ , \quad L_- \stackrel{\text{def}}{=} i(a_1^\dagger a_2 - a_2^\dagger a_1) \ , \quad L_3 \stackrel{\text{def}}{=} (a_1^\dagger a_1 - a_2^\dagger a_2) \ .$$