The Howard University
Department of Physics and Astronomy

Master of Science Comprehensive and Doctor of Philosophy Qualifying Exam

Part 1: Classical Physics

August 17, 1998

Of the seven problems, please attempt to solve *four*. Circle the *four* corresponding numbers below to indicate which of your attempted solutions should be graded. You *must* attempt to solve problem 1, and at least one out of each of the groups A and B :

1	2 3 4	5 6 7
	Group A	Group B

1. Write in your code-letter here: ☐ .
2. Place the code-letter and a page number on the *top right-hand* corner of each submitted answer sheet.
3. Write only on one side of the answer sheets.
4. Start each problem on a new answer sheet.
5. Stack your answer sheets by problem and page number, and then staple them (at the top *left-hand corner*) with this cover sheet on the top.

Good Luck!

1. A vessel of volume V_1 contains N molecules of an ideal gas held at temperature T and pressure P_1. The energy of a molecule may be written in the form:

$$E_k(p_x, p_y, p_z) = \frac{{p_x}^2}{2m} + \frac{{p_y}^2}{2m} + \frac{{p_z}^2}{2m} + \epsilon_k \ ,$$

where ϵ_k denotes the energy levels corresponding to the internal states of the molecules of gas.

[9pts] a. Evaluate the Helmholtz free energy F. Explicitly display the dependence on the volume V_1.

Now consider another vessel, also at temperature T_1, containing the same number of molecules of the identical gas held however at pressure p_2.

[7pts] b. Give an expression for the total entropy of the two gases in terms of P_1, P_2, T, N.

[9pts] c. The vessels are the connected to permit the gases to mix *without doing work.* Calculate explicitly the change in the entropy of the system. Comment on the special case when $V_1{=}V_2$ $(P_1{=}P_2)$.

Group A

2. A particle moves in a plane under the influence of a force $F = -Ar^{\alpha-1}$ directed toward the origin; A and α are positive constants. Let the potential energy be zero at the origin.

[3pts] a. Choose appropriate generalized coordinates.

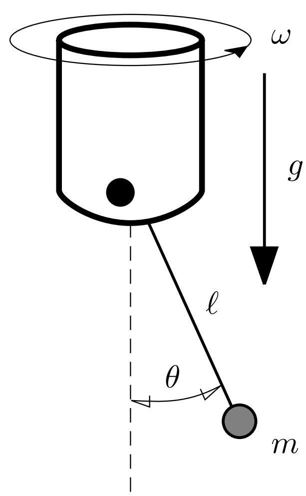
[5pts] b. Find the Lagrangian equation of motion.

[6pts] b. Is the angular momentum about the origin conserved? Show how you arrived at your answer.

[6pts] b. Is the total energy conserved? Show how you arrived at your answer.

[5pts] b. Find the Hamiltonian equations of motion.

3. A bearing of a rigid pendulum of mass m is forced to rotate uniformly with angular velocity ω. The angle between the rotation axis and the pendulum is θ. Neglect the inertia of the bearing and of the rod connecting the bearing to the bead; include the effects of the uniform force of gravity.



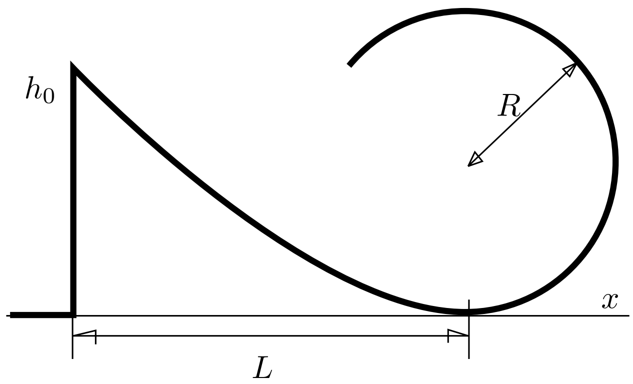
[7pts] a. Find the differential equation for $\theta(t)$.

[6pts] b. Find the angular velocity ω_c at which the stationary point at $\theta = 0$ becomes unstable.

[6pts] c. For $\omega > \omega_c$, find the stable equilibrium value of θ.

[6pts] d. Find the frequency, Ω, of small oscillations about this equilibrium value of θ.

4. A car slides without friction down a ramp described by a height function $h(x)$, which is smooth and monotonically decreasing as x increases from 0 to L. The ramp is followed by a smoothly connected loop of radius R. Gravitational acceleration is constant (g), and in the downward (negative h) direction.



[10pts] a. If the velocity is zero when $x{=}0$ (and $h{=}h_0$), find the minimum height $h_0{=}h(0)$ such that the car goes around the loop, never leaving the track.

[10pts] b. Consider the motion in the interval $0 < x < L$, before the loop. Assuming that the car always stays on the track, show that the velocity in the x-direction is related to the height as

$$\dot{x} = \sqrt{\frac{2g[h_0 - h(x)]}{1 + \left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2}} \ .$$

[5pts] c. In the particular case that $h(x) = h_0[1 - sin(\pi x/2L)]$, show that the time elapsed in going down the ramp from $(0, h_0)$ to $(L, 0)$ can be expressed as $T = (L/\sqrt{gh_0})\, f(a)$, where $a = \pi h_0/2L$, and write $f(a)$ as a definite integral. Evaluate the integral in the limiting case $h_0 \geq L$, and discuss the meaning of your answer.

Group B

5. Light of frequency ω travels through vacuum and hits, at normal incidence, the flat surface of a very good conductor of conductivity σ. Assume that $\vec{B}=\vec{H}$ everywhere, and in vacuum also $\vec{D}=\vec{E}$.

[5pts] a. Write down Maxwell's equations for a general medium, and Ohm's law.

[5pts] b. Write down plane wave expressions for $\vec{E}, \vec{H}$ and calculate $\vec{k} = \vec{k}(\omega)$.

[5pts] c. Specify the boundary conditions at the surface of the conductor.

[10pts] d. Show that the reflectivity is approximately $1 - \sqrt{\frac{2\omega}{\pi\sigma}}$ (in Gaussian units).

6. Write down Maxwell's equations for a medium with dielectric function ϵ, permeability μ, and conductivity σ.

[5pts] a. Show that there is a plane wave solution with frequency ω, and obtain an expression for the complex propagation vector $\vec{k}$.

[5pts] b. Obtain an expression for the the decay constant (skin depth) of the plane wave if this medium is a good conductor at the plane wave frequency ($\sigma \geq \omega\epsilon/4\pi$).

[5pts] c. A simple model for the conductivity of this medium can be obtained by solving the equation of motion for electrons (mass m, charge e, velocity $\vec{v}$, collision frequency γ) in a field $\vec{E}_0$, in this medium:

$$m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + mg\vec{v} = e\vec{E}_0 e^{i\omega t}$$

and use $\vec{\jmath} = en\vec{v} = \sigma\vec{E}$, where n is the electron density. Using this expression, calculate the skin depth in a tenuous plasma (γ=0) in terms of ω_p^2=$(4\pi ne^2)/m$.

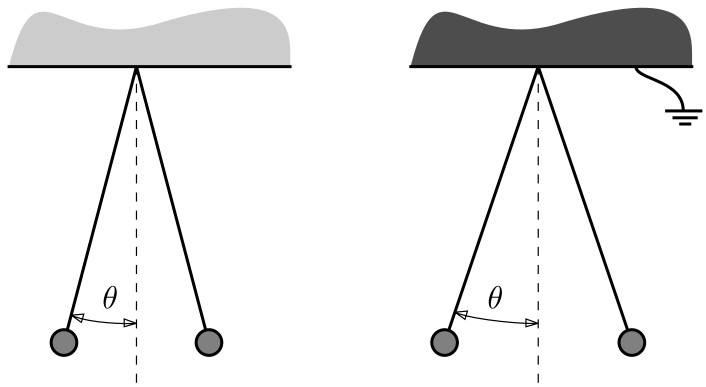
[5pts] d. In the ionosphere (a tenuous plasma), an external static magnetic induction, $\vec{B}_0$ is present, and neglecting transverse fields, the magnetic field is in the z-direction. Consider the plane wave to be circularly polarized and propagating in the z-direction also, so that E=$E_0(\hat{e}_1 + i\hat{e}_2)$. The equation of motion for the electrons in the ionosphere becomes

$$m\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} \approx e\vec{E}_\pm e^{-i\omega t} + \frac{e}{c}\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\times\vec{B}_0 \ .$$

Solve this for the displacement $\vec{x}$ and hence the polarization, to show that the index of refraction is different for left- and right-circularly polarized radiation.

[5pts] e. Find the frequency at which only one circular polarization component will propagate without attenuation, in terms of ω_p and ω_B=$\frac{eB_0}{mc}$

7. Two point charges of equal mass m and charge Q are suspended from a common point on the ceiling by two threads of length ℓ and negligible mass. Assume the angle between the two threads to be small.



[10pts] a. Find the inclination angle, θ, of each thread.

[15pts] b. Redo this problem for the same masses and charges, however with the horizontal ceiling replaced by a grounded conducting plane of infinite extent.

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Part 2: Modern Physics

August 19, 1998

Of the seven problems, please attempt to solve *four*. Circle the *four* corresponding numbers below to indicate which of your attempted solutions should be graded. You *must* attempt to solve at least one problem out of each of the three groups :

1 2	3 4 5	6 7
Group A	Group B	Group C

1. Write in your code-letter here: ☐ .
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Good Luck!

Group A

1. A relativistic particle moves with velocity $\vec{v}$ with respect to a stationary system, S.

[5pts] a. Starting with the Lorentz transformation of spacetime coordinates, obtain the velocity $\vec{v}'$ in a frame S' moving with speed u in the x-direction with respect to the frame S.

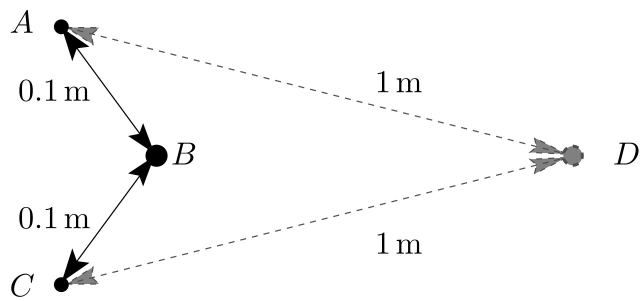
[5pts] b. Consider now a super-skateboard of length 1 m, moving with speed $\frac{3}{5}c$ with respect to the road frame S. A super-bug is racing at a speed $\frac{4}{5}c$ relative to the super-skateboard, in the same direction. In the skateboard frame, S', find the time t' for the bug to travel from one to the other end of the skateboard.

[5pts] c. Find the speed of the bug and the length of the skateboard in the Earth frame, S.

[5pts] d. In the bug's rest-frame, S'', find the time for the bug to travel from one to the other end of the skateboard.

[5pts] e. In the Earth frame, S, find the time for the bug to travel from one to the other end of the skateboard by: (i) calculating from S', and (ii) calculating from S''.

2. Charges of $5\,\mu$C are located at points A and C, as shown below.



[10pts] a. A bead of 15 g mass and $5\,\mu$C charge is released from rest at point B; calculate its speed at point D.

[15pts] b. Redo this calculation for a bead of 40×10^{-15} g mass; neglect radiative effects.

(Remark: The distance between points A and C is immaterial for answering the questions!)

Group B

3. Consider a linear harmonic oscillator, in its natural units where $\hbar$=m=k=1. Denoting $D \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}x}$, we have that the Hamiltonian is $H = \frac{1}{2}(x^2 - D^2)$, and the stationary states satisfy $H\psi_n = E_n\psi_n$. Let $A \stackrel{\text{def}}{=} (x{+}D)$ and $B \stackrel{\text{def}}{=} (x{-}D)$.

[5pts] a. Calculate the commutator $[D, x]$ and express H in terms of A and B.

[5pts] b. Calculate the commutators $[H, A]$ and $[H, B]$.

[5pts] c. Show that $A\psi_n \propto \psi_{n-1}$ and that $B\psi_n \propto \psi_{n+1}$, with $E_{n\pm1} = E_n \pm 1$.

[5pts] d. Use $A\psi_0 = 0$ to find the ground state wave-function and calculate the ground state energy E_0.

[5pts] e. Find the wave-function $B\psi_0 \propto \psi_1$ and calculate its energy, E_1.

4. Consider the effects of the perturbation $W = \Omega\,\delta(x)$ on a linear harmonic oscillator, with stationary states $|n\rangle = \sqrt{\frac{\beta}{2^n\, n! \sqrt{\pi}}} H_n(\beta x) e^{-\beta^2 x^2/2}$, with $\beta = \sqrt{m\omega/\hbar}$.

[5pts] a. Calculate the first order perturbative correction to the energies E_n for all n.

[10pts] b. Show that the second order perturbative correction to E_n is nonzero only if n is even.

[5pts] c. Extending your results so far, show that the correction to E_n—to all orders in perturbation theory—is nonzero only if n is even.

[5pts] d. Prove exactly the claim of part c., in the 'opaque barrier' limit $\Omega{\to}\infty$.

(A possibly useful relation: $\sum_{n=0}^{\infty} H_n(y) \frac{t^n}{n!} = e^{-t^2+2ty}$.)

5. A particle of mass m moves, restricted to one dimension, under the influence of the potential $V = \frac{1}{2} m v_0^2 (\frac{x}{a} - \frac{a}{x})^2$.

[5pts] a. Determine the large-x asymptotic form, $\chi(x)$, of the stationary states by keeping only the dominant terms in the Schrödinger equation and solving it to leading order.

[5pts] b. Determine the small-x asymptotic form, $\phi(x)$, of the stationary states by keeping only the dominant terms in the Schrödinger equation and solving it to leading order.

[7pts] c. For small a, consider this to be a perturbation of the linear harmonic oscillator of frequency $\omega = \frac{v_0}{a}$. Write down the general expression for the first order correction to the energy levels and prove that (half of) these and higher order corrections diverge.

[8pts] d. Explain why perturbation theory fails regardless of how small a is.

Group C

6. Consider the 2p states in the Hydrogen atom; $a_0 = \hbar^2/m_e e^2 = 0.52\,\text{Å}$ is the Bohr radius.

[3pts] a. Specify the values of the principal and orbital quantum numbers, n, ℓ and list the allowed values of the magnetic quantum number, m.

[6pts] b. Using $\psi_{2,1,0} = \frac{1}{\sqrt{32\pi {a_0}^5}}\, r\cos\theta\, e^{-r/2a_0}$, find the probability density for the 2p electron to be found with m=0. Sketch separately the radial and the θ-dependent factor.

[6pts] c. Using $\psi_{2,1,\pm1} = \frac{1}{\sqrt{64\pi {a_0}^5}}\, r\sin\theta \mathrm{e}^{\pm i\varphi}\, \mathrm{e}^{-r/2a_0}$, find the probability density for the 2p electron to be found with $|m|$=1. Sketch separately the radial and the θ-dependent factor.

[5pts] d. Determine the probability that the 2p electron is found in the spherical shell between R and R+dr.

[5pts] e. Calculate $\langle 2, 1, m'|z|2, 1, m\rangle$ for all allowed values of m, m'.

7. Two electrons are confined to linear motion, and move freely within $0 < x_1, x_2 < L$.

[5pts] a. Write down the ground state wave-function $\psi(x_1, x_2)$ for the two-electron system if it is known that the total spin is S=0.

[7pts] a. Calculate the probability that both electrons (with S=0) are found in the same half of the interval, say $0 < x_1, x_2 < \frac{1}{2}L$.

[5pts] c. Write down the ground state wave-function $\psi(x_1, x_2)$ for the two-electron system if it is known that the total spin is S=1.

[8pts] d. Calculate the probability that both electrons (with S=1) are found in the same half of the interval, say $0 < x_1, x_2 < \frac{1}{2}L$.