The Howard University
Department of Physics and Astronomy

Master of Science Comprehensive and Doctor of Philosophy Qualifying Exam

Part 1: Classical Physics

August 22, 2001

Work out the solutions to *four* problems,– at least one from each group. Circle the numbers below to indicate your choice of problems.

Group A	Group B	Group C
1 2 3	4 5	6

1. Write in your code-letter here: ☐ .
2. Write your code-letter and a page number (in sequential order) on the *top right-hand* corner of each submitted answer sheet.
3. Write only on one side of the answer sheets.
4. Start each problem on a new answer sheet.
5. Stack your answer sheets by problem and page number, and then staple them (at the top *left-hand corner*) with this cover sheet on the top.

Good Luck!

Group A

1. Consider a projectile of mass m moving in a region where air resistance is a function of velocity, given by $\vec{F}_{\rm air} = -b\,\vec{v}$. The projectile starts from the origin, with the initial velocity $\vec{v}_0 = v_{0x}\hat{\rm e}_x + v_{0z}\hat{\rm e}_z$ (see figure below).

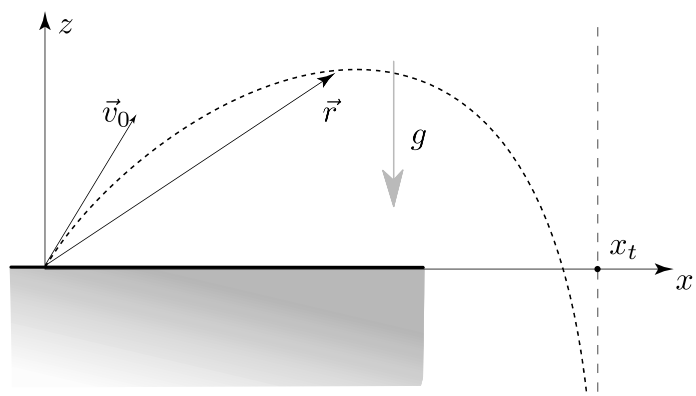
[5pts] *a.* Use Newtonian mechanics and derive an expression for the x-component of the velocity vector, $\vec{v}(t)$.

[5pts] *b.* Determine the expression for the x-component of the position vector $\vec{r}(t)$.

[5pts] *c.* Derive the expression for the z-component of the velocity, $\vec{v}(t)$.

[5pts] *d.* Determine the expression for the z-component of the position vector $\vec{r}(t)$.

[5pts] *e.* For large values of t, the projectile approaches a vertical asymptote (see figure), and moves at a terminal velocity. Show that the vertical asymptote is given by the expression $x_t = m\,v_{0x}/b$ and that the terminal velocity is given by $\vec{v}_t = -\frac{mg}{b}\hat{\rm e}_z$.



2. Suppose that the solar system is immersed in a uniformly dense spherical cloud of weakly interacting massive particles (WIMPs). Then, objects inside the solar system would experience gravitational forces from both the Sun and the cloud of WIMPs, such that $F_r = -k\,r^{-2} - b\,r$. Assume that the extra force due to WIMPs is very small, *i.e.*, that $b \ll k\,r^{-3}$ for distances within the solar system.

[10pts] *a.* Find the frequency of radial oscillations for a nearly circular orbit.

[15pts] *b.* Describe the shapes of the orbits when r is large enough, so that $F_r \approx -b\,r$, *i.e.*, when the Sun's gravitational force is negligible as compared to the one dues to WIMPs.

Group A

3. Consider a 2-dimensional harmonic oscillator in two dimensions, with generalized coordinates q_σ, $\sigma = 1, 2$ and two different spring constants, k_1, k_2.

[2pts] *a.* Write down the expression for the Hamiltonian of the system, assuming that the forces involved are conservative.

[4pts] *b.* What is the Hamilton-Jacobi equation for the Hamilton's characteristic function $W(q_1, q_2)$?

[5pts] *c.* Assuming that $W(q_1, q_2) = W_1(q_1) + W_2(q_2)$, use separation of variables to obtain separate differential equations for W_1 and W_2, respectively. (Define separation constants α_1, α_2 so that the total energy is $\alpha = \alpha_1 + \alpha_2$.)

[5pts] *d.* Obtain the canonical momenta p_σ, $\sigma = 1, 2$.

[4pts] *e.* Introduce a new variable θ related to q_σ by $q_\sigma = \sqrt{\frac{2\alpha_\sigma}{k_\sigma}} \sin\theta$ and obtain the action integral J_σ.

[5pts] *f.* Determine the fundamental frequencies ν_σ, $\sigma = 1, 2$ of the independent modes of motion of the 2-dimensional oscillator.

Group B

4. Consider a rectangular conducting box of sizes a, b, c in the x, y, z direction, respectively. All sides except the bottom one are grounded (held at zero potential) and the bottom one is held at the potential $\Phi_0 = const \neq 0$, and insulated from the rest.

[5pts] *a.* List all the boundary conditions which restrict the electrostatic potential, $\Phi(x, y, z)$.

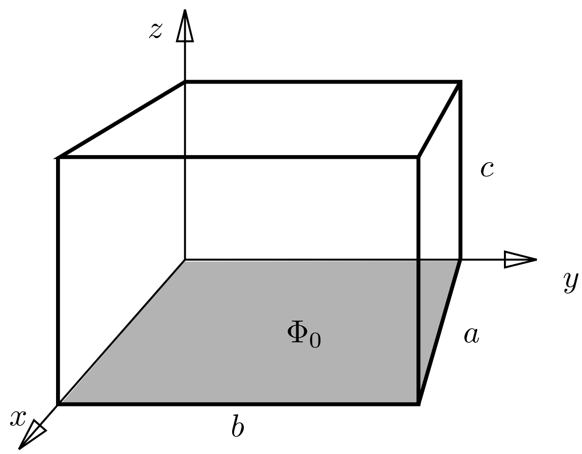
[5pts] *b.* Calculate $\Phi(x, y, z)$ inside the box.

Let $\Phi_1(x, y, z) = \Phi(x, y, z; a, b, c, \Phi_0)$ denote the solution of part b. A charge q is now placed in the center of the box, at $x = a/2$, $y = b/2$, $z = c/2$.

[10pts] *c.* Determine the resulting potential inside the box.

[5pts] *e.* Assume now that you have a conducting box as in parts a–b., with the same boundary conditions, except that now the side in the x, z-plane is set at the potential Φ_1. Express the potential inside this box using $V(x, y, z; a, b, c, \Phi_0)$, the solution of part b.

(Hint: in parts c. and d., you need not write the solution of part a. in detail.)



5. An electron is released, from being held at rest at a large (but finite) distance, R, from a nucleus of charge Ze, so that it "falls" toward the nucleus.

[5pts] *a.* Calculate the angular distribution of the emitted radiation (Poynting vector).

[5pts] *b.* Calculate the polarization of the emitted radiation.

[8pts] *c.* Calculate the radiated power as a function of the separation between the electron and the nucleus.

[7pts] *d.* Calculate the total energy radiated between distances r_1 and r_2 (with $r_2 < r_1$).

Assume that $v \ll c$ at all times and neglect the radiative reaction force, so that the (retarded) radiation field is $\vec{E}_{\rm rad} = -\frac{|e|}{c^2}\left[\frac{\hat{n}\times(\hat{n}\times\dot{\vec{v}})}{r}\right]$ at $t' = t - \frac{r}{c}$ and energy is conserved.

Group C

6. The Carnot cycle consists of a sequence of four reversible processes, alternating between isothermal and adiabatic ones. An ideal gas is usually taken as the working substance. Suppose that a Carnot cycle is constructed instead using photon gas, for which we know all the thermodynamic functions, such as U, F, S, P, are valid at all temperatures. It has been established that the energy density of the photon gas is $u = \sigma T^4$, where σ is the Stefan-Boltzmann constant.

[5pts] *a.* Find the specific heat at constant volume and the entropy for the photon gas.

[5pts] *b.* Find the pressure of the photon gas, and show that it depends only on its temperature.

[4pts] *c.* Show that the equation of state for the photon gas is given by $PV^\beta = f(S)$, where $\beta = 4/3$ and $F(S)$ is a function of entropy only.

[3pts] *d.* Sketch the Carnot cycle for the photon gas in the P-V plane.

[3pts] *e.* By considering heat exchange in the processes, calculate the Carnot efficiency and show that the photon gas engine yields the same efficiency as the conventional ideal gas engine.

[5pts] *f.* By calculating the work done for each of the four processes, determine the total work done per cycle for the photon gas engine, and show that it equals the net heat absorbed per cycle.

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Part 2: Modern Physics

August 24, 2001

Work *four* problems, at least one from each group. Circle the numbers below to indicate your chosen problems.

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Good Luck!

Group A

1. Consider the action of infinitesimal rotations and Lorentz boosts, $(x, y, z, t) \to (x', y', z', t')$, on a scalar function $f(x, y, z, t)$. A rotation about the z-axis by an infinitesimal angle, θ_z, induces

$$x' \approx x - \theta_z\, y\ , \quad y' \approx y + \theta_z\, x\ , \quad z' = z\ , \text{ and } t' = t\ .$$

Then, $f(x', y', z', t') - f(x, y, z, t) = \theta_z(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})f(x, y, z, t) = \frac{i}{2}\theta_z\, \hat{J}_z\, f(x, y, z, t)$ defines the generator of z-rotations, $\hat{J}_z = -2i(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$.

[5pts] *a.* In a similar fashion, write down the effect of a Lorentz boost in the z-direction by an infinitesimal velocity, v_z, on Cartesian coordinates, and calculate the generator of z-boosts, $\hat{L}_z$, defined by $f(x', y', z', t') - f(x, y, z, t) = \frac{i}{2}v_z\, \hat{L}_z\, f(x, y, z, t)$.

[6pts] *b.* Obtaining $\hat{J}_x, \hat{J}_y$ from $\hat{J}_z$ by cyclic permutations, calculate $[\hat{J}_i, \hat{J}_j]$ for $i, j = x, y, z$.

[7pts] *c.* Obtaining $\hat{L}_x, \hat{L}_y$ from $\hat{L}_z$ by cyclic permutations, calculate $[\hat{L}_i, \hat{L}_j]$ for $i, j = x, y, z$.

[7pts] *d.* Finish obtaining the Lorentz algebra by calculating $[\hat{J}_i, \hat{L}_j]$ for $i, j = x, y, z$.

2. Consider the decay, $A \to B + C$, of a particle with rest mass m_A into two particles with rest masses m_B and m_C.

[6pts] *a.* If A is at rest in the lab frame before the decay, show that the lab frame total energy of the particle B is $E_B = \frac{1}{2m_A}(m_A^2 + m_B^2 - m_C^2)$.

[7pts] *b.* If A decays while moving at a constant velocity with respect to the lab, find the relation between the total energies, E_A, E_B, and the angle, θ_{AB}, between the direction of motion of A before the decay and that of B after the decay.

[7pts] *c.* An atom of rest mass M decays into a state of rest energy $M - \delta$ by emitting a photon of energy $h\nu$. Show that $h\nu < \delta$.

[3pts] *d.* In the Mössbauer effect, the recoil is absorbed by a macroscopic piece of material. Show that this implies that now $h\nu \approx \delta$.

Group B

3. Consider a particle of charge q and mass m in constant crossed $\vec{E}$ and $\vec{B}$ fields:

$$\vec{E} = (0, 0, E) \ , \qquad \vec{B} = (0, B, 0) \ , \qquad \vec{r} = (x, y, z) \ .$$

[3pts] *a.* Determine the scalar and vector potentials, $\Phi, \vec{A}$, for this field in a gauge where $\vec{A}$ has a single Cartesian component.

[5pts] *b.* Write the Schrödinger equation for this particle, in this gauge.

[7pts] *c.* Separate variables and reduce it to an effective 1-dimensional problem.

[3pts] *d.* Compare this to a linear harmonic oscillator and calculate the expectation value of $\vec{r}$.

[7pts] *e.* Calculate the expectation value of the velocity, $\vec{v}$, in any energy eigenstate, using Ehrenfest's theorem: $\frac{\mathrm{d}\langle Q\rangle}{\mathrm{d}t} = \langle\frac{\partial Q}{\partial t}\rangle - \frac{1}{i\hbar}\langle[H, Q]\rangle$.

4. A particle of mass m and electric charge q is constrained to move on a ring of radius R.

[5pts] *a.* Write down the Schrödinger equation for this particle, neglecting any radiation losses. Determine its energy levels and their degeneracy, if any.

An *infinitely* long solenoid of radius $a \ll R$ and (inside) magnetic flux $\Phi = \pi\, a^2 B$ is placed along the axis of rotational symmetry of the ring.

[5pts] *b.* Determine the magnetic field, and the vector and scalar potentials *outside* the solenoid, in the 'radiation' gauge ($\vec{\nabla}\cdot\vec{A}$=0).

[5pts] *c.* Write down the Schrödinger equation for the particle orbiting the solenoid.

[5pts] *d.* Solve this equation: determine all energy levels and corresponding wavefunctions.

[5pts] *e.* Treating Φ as a small and tunable parameter, explain what happened to the degeneracy of the energy levels observed in part a.

Group C

5. A particle of mass m is trapped in an $L \times L$ square, but moves freely within $0 \leq x \leq L$ and $0 \leq y \leq L$.

[5pts] *a.* Write down the Schrödinger equation and the boundary conditions for this particle.

[5pts] *b.* List the energy levels and the corresponding wavefunctions, and state the degeneracy of the lowest four energy levels.

[10pts] *c.* Determine the effect of the perturbation $H' = \lambda\, \delta(x - \frac{L}{2})$ on the energies of the states with unperturbed energy $5\,\frac{\pi^2\hbar^2}{2mL^2}$.

[5pts] *d.* Derive a boundary (matching) condition for the wavefunction at $x = \frac{L}{2}$, induced by the perturbation, and specify the class of states which will not be affected by it.

6. Develop a variational principle from the Schrödinger equation

$$-\frac{\hbar^2}{2m}\vec{\nabla}^2\psi_n + V(\vec{r})\psi_n = E_n\psi_n \ ,$$

where ψ_n is the wave-function of the stationary state with energy E_n, and $V(\vec{r})$ is a non-negative real potential.

[5pts] *a.* Using that $\int \mathrm{d}^3\,\vec{r}\psi_n^*\psi_{n'} = \delta_{n,n'}$, obtain an integral expression for E_n, in terms of ψ_n, $\vec{\nabla}\psi_n$, their conjugates and $V(\vec{r})$, but no higher derivatives. Show that $E_n \geq 0$.

[7pts] *b.* Vary this integral expression for E_n with respect to a small change $\psi_n \to \psi_n + \delta\psi$, and determine δE_n as an integral expression quadratic in $\delta\psi$ and $\vec{\nabla}\psi_n$.

[10pts] *c.* Using the completeness of the set $\{\psi_n\}$, expand $\delta\psi = \sum_k c_k\psi_k$ and integrate the expression for δE_n to obtain δE_n as a function of c_k, c_k^*, E_k and E_n.

[3pts] *d.* Which ψ_n (*i.e.*, which E_n) can be determined by minimizing δE_n?

Note: n, n', k need not be integers or even discrete variables.