The Howard University
Department of Physics and Astronomy

Master of Science Comprehensive and Doctor of Philosophy Qualifying Exam

Part 1: Classical Physics

August 19, 2002

Work out the solutions to *four* problems,– at least one from each group. Circle the numbers below to indicate your choice of problems.

1 2	3 4	5 6 7
Group A	Group B	Group C

1. Write in your code-letter here: ☐ .
2. Write your code-letter and a page number (in sequential order) on the *top right-hand* corner of each submitted answer sheet.
3. Write only on one side of the answer sheets.
4. Start each problem on a new answer sheet.
5. Stack your answer sheets by problem and page number, and then staple them (at the top *left-hand corner*) with this cover sheet on the top.

Good Luck!

Group A

1. Consider an ensemble of N classical, indistinguishable particles with one degree of freedom, in thermal equilibrium with a large thermal reservoir at temperature T. The energy levels of each particle are $E_j = (4j + 3)E_0$, for $j = 0, 1, 2$.

[5pts]   *a.* Using the appropriate statistics, state the equilibrium probability, P_j, of finding a particle with energy E_j.

[7pts]   *b.* Calculate the partition function, Z, for this system.

[6pts]   *c.* Calculate the Helmholtz free energy, F, for this system.

[7pts]   *d.* Calculate the internal energy, U, for this system.

2. In a particular model of thermal behavior of crystalline solids, each of the N atoms of the solid behaves like *three independent* harmonic oscillators. These $3N$ harmonic oscillators (which are on *distinguishable* sites) all have the same angular frequency of vibration, ω_0, and their possible energy levels are given as $E_n = \hbar\omega_0(n + \frac{1}{2})$, where $n = 0, 1, 2, \cdots$

[6pts]   *a.* Calculate the Helmholtz free energy F of this solid.

[7pts]   *b.* Calculate the heat capacity, C, as a function of temperature, T. Determine the general formula and, to leading order, calculate the low-temperature limit and the high-temperature limit.

[6pts]   *c.* Sketch a plot of C *vs.* T.

[6pts]   *d.* Calculate the average value, $\bar{n}(T)$, of n.

Group B

3. A $h_0 = 2\,\text{m}$ tall man is bungee-jumping from a platform situated at a height $h = 25\,\text{m}$ above a river. One end of an elastic rope is attached to his foot and the other end is fixed to the platform. He starts falling from rest, in a vertical position. The length and elastic properties of the rope are chosen so that his speed will have been reduced to zero at the instant when his head reaches the surface of the water. Ultimately, the jumper is hanging from the rope, with his head $8\,\text{m}$ above the water. The rope is massless and has a spring constant k. You may also assume that the man's mass is evenly distributed along the length of his body.

[9pts]　　*a.* Calculate the unstretched length of the rope, ℓ_0.

[7pts]　　*b.* Calculate the man's maximum speed during the jump.

[9pts]　　*c.* Calculate the man's maximum acceleration during the jump.

4. The mass, m, of a spring is uniformly distributed along its length, and it has a spring constant k. A bead of mass M is hung at the end of the spring and then set into purely vertical oscillation.

[8pts]　　*a.* Carefully ascertaining the motion of both the bead and the spring, determine the kinetic energies of both parts.

[5pts]　　*b.* Determine the Lagrangian for this particular system.

[7pts]　　*c.* Derive the equation of motion for the bead.

[5pts]　　*e.* Calculate the period of oscillation of the bead.

　　(Hint: the period should be between $2\pi\sqrt{M/k}$ and $2\pi\sqrt{(M+m)/k}$)

Group C

5. The classical Hamiltonian for a particle of mass m and charge q in the presence of electric and magnetic fields is $H = \frac{1}{2m}(\vec{p} - \frac{q}{c}\vec{A})^2 + q\Phi$.

[5pts] *a.* Determine the electric and magnetic vector fields, $\vec{E}$ and $\vec{B}$, in terms of the quantities given above.

[6pts] *b.* From the relations for $\vec{E}$ and $\vec{B}$ and Maxwell's equations, show that the potentials, $\vec{A}$ and Φ, satisfy the Lorentz condition, $\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0$.

[8pts] *c.* If the potentials are subject to the gauge transformation, $\vec{A} \to \vec{A} + \vec{\nabla}\Lambda$ and $\Phi \to \Phi - \frac{\partial\Lambda}{\partial t}$, determine the condition on Λ for the Lorentz condition to remain unchanged, and determine the induced gauge transformations of $\vec{E}$, and $\vec{B}$.

[7pts] *d.* Starting from the Lorentz force, derive its corresponding generalized potential, and use this to derive the Lagrangian and then the above Hamiltonian.

6. A thin superconducting ring (of zero resistance, mass $50\,\text{mg}$, radius $r_0 = 0.5\,\text{cm}$ and inductance $L = 1.3\times10^{-8}\,\text{H}$) is held coaxially above a vertical, cylindrical magnetic rod. The radius of the magnetic rod is much larger than that of the ring. While the ring is held at rest, there is no current flowing through it. When it is released, it starts to fall downward, keeping its axis collinear with that of the magnet. The magnetic field produced by the rod is given by $\vec{B} = B_0\big(\beta r\hat{r} + (1 - \alpha z)\hat{z}\big)$, where $B_0 = 0.01\,\text{T}$, $\alpha = 2\,\text{m}^{-1}$ and $\beta = 32\,\text{m}^{-1}$.

[5pts] *a.* Calculate the total flux through the superconducting ring.

[5pts] *b.* Explain why the total flux through the ring must remain constant.

[4pts] *c.* Derive the equation of motion for the ring's vertical position, $z(t)$, and determine the numerical values of the amplitude and frequency.

[3pts] *d.* Derive an expression for the total current flowing in the ring as a function of time.

[3pts] *e.* Describe the direction of the current flow in the ring as a function of time.

7. Two polar molecules, each with the electric dipole moment $\vec{p}_1$ and $\vec{p}_2$, respectively, are separated by a distance r. The dipole moments are orthogonal to each other: $\vec{p}_1$ is vertical, pointing upward and $\vec{p}_2$ is horizontal, pointing away from $\vec{p}_1$.

[8pts] *a.* Calculate the force (magnitude and direction) on $\vec{p}_2$ due to $\vec{p}_1$'s electrostatic field.

[8pts] *a.* Calculate the force (magnitude and direction) on $\vec{p}_1$ due to $\vec{p}_2$'s electrostatic field.

[9pts] *a.* Calculate the total torque (magnitude and direction) exerted on $\vec{p}_2$ with respect to the center of $\vec{p}_1$.

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Part 2: Modern Physics

August 21, 2002

Work *four* problems, at least one from each group. Circle the numbers below to indicate your chosen problems.

$$\boxed{1 \ 2 \ 3} \ \boxed{4 \ 5} \ \boxed{6 \ 7}$$

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Good Luck!

Group A

1. Use Newton's second law of motion and Coulomb's law to describe the Hydrogen atom, assuming that its electron moves in a circular orbit of radius r centered at the proton.

[6pts]   *a.* Calculate the kinetic and the potential energy of the electron-proton system.

[7pts]   *b.* Express r in terms of the quantum number n using Bohr's selection rule for the angular momentum of the electron, and show that its total energy is $E_n = -\frac{\mathbf{R}}{n^2}$, where $\mathbf{R} = \frac{m_e e^4}{2(4\pi\epsilon_0\hbar)^2}$ and $n = 1, 2, 3, \cdots$ is the principal quantum number.

[7pts]   *c.* For transitions between the ground state $(n = 1)$ and the lowest five excited states $(1 < n \le 6)$, compute the wavelengths of the Hydrogen emission lines (Lyman series).

[5pts]   *d.* Consider transitions between near states near the free electron limit $(n \gg 1$, while $\triangle n = 1, 2, 3$ and $\triangle n \ll n)$, and show that the frequencies of the emitted/absorbed photons approach the classical values, $\omega_{\text{cl.}} = k\Omega_n$, where $k = 1, 2, 3$ and $\Omega_n = \frac{2\mathbf{R}}{n^3\hbar}$ (Bohr's complementarity principle).

2. A free electron at rest, absorbs a $\lambda = 0.071\,\text{nm}$ X-ray photon, incident along the x-axis.

[4pts]   *a.* Show that the existence of this resulting ("excited") electron $(\tilde{\text{e}}^-)$ violates the strict conservation of relativistic 4-momentum.

[5pts]   *b.* Use the difference in the square of the magnitude of the momentum of $\bar{\text{e}}^-$ between the value predicted by conservation of (relativistic) energy *vs.* conservation of momentum to estimate the uncertainty in the momentum of $\tilde{\text{e}}^-$. From this, estimate a position and time uncertainty for $\tilde{\text{e}}^-$.

[6pts]   *c.* Within the time of part *b.*, $\tilde{\text{e}}^-$ emits a photon of wavelength λ', at an angle θ with respect to the x-axis; the electron scatters, in the direction opposite from the photon, at an angle ϕ from the x-axis. Derive the Compton formula, relating $\triangle\lambda \overset{\text{def}}{=} \lambda'-\lambda$, θ and fundamental constants.

[10pts]   *d.* If the photon and the electron are known to have scattered at 90° from each other, derive a relation between the photon's scattering angle, θ, and the ratio λ'/λ, and the relation between λ, θ and fundamental constants.

   $m_e = 9.109 \times 10^{-31}\,\text{kg}, c = 3 \times 10^8\,\text{m/s}, \hbar = 1.055 \times 10^{-34}\,\text{Js}.$

3. Consider the α-decay of a nucleus by describing tunneling through a spherically symmetric barrier $V(r)$, which is assumed to be very steep at the classical turning points, $R \ll D$.

[6pts]   *a.* Write down the Schrödinger equation, simplify it by setting $\ell = 0$ (for S-waves) and $\psi(r) = r^{-1}u(r)$, and state the appropriate boundary conditions.

[7pts]   *b.* Assuming that $V(r)$ is approximately zero inside $(0 < r < R)$ and outside $(r > D)$ the barrier, write down the expected general form of the wave-function in these regions as appropriate for α-decay. Using a semiclassical (WKB) estimate for the decaying $\psi(r)$ inside the barrier, derive the estimate for the transmission coefficient in the standard form, $T = e^{-2\gamma}$, with $\gamma = \int_R^D \mathrm{d}r \sqrt{2m_\alpha(V(r) - E)/\hbar^2}$.

[7pts]   *c.* Realistically, the barrier is provided by the Coulomb repulsion between the twice positively charged α-particle and the $Z' \stackrel{\text{def}}{=} (Z{-}2)$-charged daughter nucleus. Use this to determine the barrier, $V(r)$ for $R < r < D$, where $R = const.$, D satisfies $V(D) = E$, and then integrate γ.

[3pts]   *d.* Expand the result for γ to second order in $\frac{R}{D} \ll 1$, determine and state the dependence of transmission coefficient on Z.

(Useful results: $\int_a^b \mathrm{d}x \sqrt{b/x - 1} = b \cos^{-1} \sqrt{a/b} - \sqrt{a(b - a)}$, and $\cos^{-1}(\epsilon) \approx \frac{1}{2}\pi - \epsilon - \frac{1}{6}\epsilon^3$.)

Group B

4. Consider an electron (of mass m_e) trapped but otherwise moving freely within a horizontal $L \times (L/\sqrt{3})$ rectangle such that $x \in [0, L]$ and $y \in [0, L/\sqrt{3}]$.

[4pts] *a.* Write down the Hamiltonian, the boundary conditions for this electron, the unnormalized wave-functions, $\psi_{n_x,n_y}(x, y)$, and their energies, E_{n_x,n_y}, for this electron.

[3pts] *b.* Calculate the normalization constants for the wave-functions.

[6pts] *c.* List the 10 states with energy less than $30\frac{\pi^2\hbar^2}{2m_e L^2}$, and state their degeneracy.

[6pts] *d.* With the system in its ground state, the wall at $y = L/\sqrt{3}$ is suddenly moved to $y = L$. Specify and sketch the wave-function of the system immediately after the change. Calculate the probability that the system will be found in the new ground state immediately after the change.

[6pts] *e.* Calculate the shift in the energy, to first order in perturbation theory, due to the constant weak electric field $\vec{\mathcal{E}} = \mathcal{E}_0 \hat{e}_x$, and state whether this removes any degeneracy.

Useful results: $\sin(a)\sin(b) = \frac{1}{2}\big(\cos(a{-}b) - \cos(a{+}b)\big)$, $\int_0^\pi d\phi \sin^2(n\phi)\,\phi = \frac{\pi^2}{4}$.

5. Consider a 3-dimensional harmonic oscillator, with the position and linear momentum operators given in a Cartesian basis as $\hat{\vec{r}} = \hat{e}_j \hat{x}_j$ and $\hat{\vec{p}} = \hat{e}_j \hat{p}_j$, respectively. The canonical commutation relations (CCRs) may then be stated as $[\hat{x}_j, \hat{p}_k] = i\hbar\delta_{jk}$.

[5pts] *a.* By direct calculation, prove that $[\hat{x}_j, \hat{L}_k] = i\hbar\epsilon_{jkl}\hat{x}_l$ and $[\hat{p}_j, \hat{L}_k] = i\hbar\epsilon_{jkl}\hat{p}_l$, for all $j, k = 1, 2, 3$.

[5pts] *b.* By direct calculation, prove that $\hat{r}^2$, $\hat{p}^2$, $\hat{\vec{r}}\cdot\hat{\vec{p}}$, and $\hat{\vec{p}}\cdot\hat{\vec{r}}$ all commute with $\hat{\vec{L}}$, and explain why this must be true.

[10pts] *c.* Using that $\hat{\vec{L}}\times\hat{\vec{p}} = \hat{e}_j\epsilon_{jkl}\hat{L}_k\hat{p}_l$ and the above results, show that $[\hat{H}, \hat{\vec{K}}_0] = -2\hbar\hat{\vec{r}}$, for $\hat{\vec{K}}_0 \stackrel{\text{def}}{=} \frac{1}{2}\big(\hat{\vec{L}}\times\hat{\vec{p}} + \hat{\vec{p}}\times\hat{\vec{L}}\big)$ and $\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{r}^2$.

[5pts] *d.* Determine the constants α, β in $\hat{\vec{K}} = \hat{\vec{K}}_0 + \alpha\hat{\vec{r}} + \beta\hat{\vec{p}}$ such that $\hat{\vec{K}}$ is a conserved quantity.

(Hint: Working in Cartesian components, index notation and using only the CCRs will simplify calculations a great deal.)

Group C

6. Consider a two-level system with eigenstates $|1\rangle$ and $|2\rangle$ of the Hamiltonian, having energies ϵ_1 and ϵ_2, respectively, where $\epsilon_2 > \epsilon_1$.

[6pts]   *a.* If the system is in the state $|\psi\rangle = a|1\rangle + b|2\rangle$, write down the density matrix ρ.

Let the system be in thermal equilibrium at temperature T.

[7pts]   *b.* Calculate the density matrix, ρ, as a function of T.

[5pts]   *c.* Calculate the Helmholtz free energy, F, as a function of T.

[7pts]   *d.* Calculate the entropy, S, as a function of T.

7. Consider the elastic scattering of a particle of mass m and incident wave-vector $\vec{k}$ from a potential which is specified as a function of the wave-vector transfer, $\vec{q} \stackrel{\text{def}}{=} (\vec{k}' - \vec{k})$, by $t(\vec{q}) \stackrel{\text{def}}{=} \langle \vec{k}'|V|\vec{k}\rangle = -A(\alpha + i)e^{-\beta q^2}$; here $|\vec{k}\rangle$ and $|\vec{k}'\rangle$ are the incident and the scattered (asymptotic) plane wave states.

[5pts]   *a.* Derive the relation $|\vec{q}| = 2|\vec{k}|\sin(\frac{\theta}{2})$, where θ is the scattering angle.

[7pts]   *b.* Derive a relation between the scattering amplitude, $f(\theta)$ and $t(\vec{q})$.

[10pts]   *c.* State the optical theorem and use it to calculate the total cross-section.

[3pts]   *d.* Does the total cross-section equal the elastic cross-section for this problem? Explain.

Hint: you may use Fermi's "golden rule" and the standard relation between $f(\theta)$ and $\frac{d\sigma}{d\Omega}$.