# The Howard University
# Department of Physics and Astronomy

## Master of Science Comprehensive and Doctor of Philosophy Qualifying Exam

Part 1: Classical Physics

August 26, 2003

Work out the solutions to *four* problems,– at least one from each group. Circle the numbers below to indicate your choice of problems.

| 1 2 3 | 4 5 | 6 7 |
|---|---|---|
| Group A | Group B | Group C |

**1.** Write in your code-letter here: ☐ .
**2.** Write your code-letter and a page number (in sequential order) on the *top right-hand* corner of each submitted answer sheet.
**3.** Write only on one side of the answer sheets.
**4.** Start each problem on a new answer sheet.
**5.** Stack your answer sheets by problem and page number, and then staple them (at the top *left-hand corner*) with this cover sheet on the top.

Good Luck! (And may you not need it.)

## Group A

**1.** In the linear molecule of $CO_2$, the atom of Carbon (with mass $M$) is flanked on both sides by an atom of Oxygen (with mass $m$). The C-O bonding force may be approximated by a harmonic force with a spring constant $k$.

[5pts] *a.* Determine the Lagrangian for small vibrations along the O-C-O axis.

[5pts] *b.* Derive the equations of motion.

[6pts] *c.* Determine the frequencies of the normal modes.

[5pts] *d.* Determine, sketch and describe the normal modes.

[4pts] *e.* Sketch two planar but non-collinear normal modes of this molecule.

**2.** Consider a system with one degree of freedom (and generalized coordinate $q$) the dynamics of which is governed by the potential $V(q)$ with an equilibrium at $q$=0.

[6pts] *a.* Determine the frequency of small oscillations about $q$=0.

[7pts] *b.* A massless elastic cord (spring constant $k$) is fixed at its endpoints, spanning a horizontal distance of $2\ell$. A block of mass $m$ is then suspended from the middle of the cord, causing it to sag the vertical distance $y$. Determine the potential energy for the system.

[6pts] *c.* Show that the equilibrium position of the system is given by a root of the equation

$$u^4 - 2au^3 + a^2u^2 - 2au + a^2 = 0 ,$$

where $u = y/\ell$; determine $a$ in terms of $m, g, k, \ell$.

[6pts] *d.* Determine the frequency of vertical small oscillations about the equilibrium position of this system.

**3.** A straight tunnel is dug from New York to San Francisco, which are on the same latitude ($\approx 40°$) and 5,000 km apart, when measured along the surface. A car rolling on steel rails is released from rest at New York, and rolls through the tunnel to San Francisco.

[8pts] *a.* Neglecting friction and also the rotation of the Earth, calculate the time it will take for the car to complete the one-way trip. Use $g$=9.80 m/s$^2$ and $R_E = 6,400$ km.

[8pts] *b.* Including a decelerating friction force $F_f = -\alpha\dot{x}|\dot{x}|$, with $\alpha > 0$, but ignoring the effects of Earth's rotation, derive the differential equation for the phase-space trajectory.

[9pts] *c.* Ignoring now friction, estimate the magnitude of the centrifugal and Coriolis forces relative to the gravitational force.

## Group B

**4.** A small electric dipole (of moment $\vec{p}$) is suspended above an infinite and perfectly conducting plane at a height $h$. Use the method of images to analyze it.

[6pts] *a.* For $\vec{p}$ perpendicular to the plane, determine the force exerted on the dipole. Is it attractive? Does it depend on the orientation of $\vec{p}$?

[6pts] *b.* For $\vec{p}$ parallel to the plane, determine the force exerted on the dipole. Is it attractive?

[7pts] *c.* For general $\vec{p}$, determine the force exerted on the dipole. Is it attractive?

[6pts] *d.* Calculate the energy required to move the dipole to infinity.

**5.** A long coaxial cable consist of concentric regions: (1) the wire, $r < a$, (2) insulation, $a < r < b$, the shielding, $b < r < c$, and insulation on the outside. A current $I$ is flowing along the central wire and returning through the shielding.

[6pts] *a.* Find the magnetic field, $\vec{B}$, for $r < a$.

[6pts] *b.* Find the magnetic field, $\vec{B}$, for $a < r < b$.

[6pts] *c.* Find the magnetic field, $\vec{B}$, for $b < r < c$.

[7pts] *d.* Find the self-inductance of a length $\ell$ of this cable.

## Group C

**6.** Consider the temperature, $T$, and the volume, $V$, of a homogeneous substance to be independent.

[5pts] *a.* Using the definition of heat capacity at constant volume, $C_V$, and a Maxwell relation, derive:

$$\Big(\frac{\partial C_V}{\partial V}\Big)_T = T\Big(\frac{\partial^2 p}{T^2}\Big)_V \ .$$

[5pts] *b.* Writing $p$ for the pressure and $U$ the internal energy of the same substance, derive:

$$\Big(\frac{\partial U}{\partial V}\Big)_T = T\Big(\frac{\partial p}{T}\Big)_V - p \ .$$

[5pts] *c.* For a gas with molar volume $v$ and governed by the state equation $(p+\frac{a}{v^2})(v-b) = RT$, prove that the molar specific heat capacity at constant volume, $c_v$, is independent of the volume and so may depend only on the temperature.

[5pts] *d.* If the gas is taken from a standard state $(T_0, v_0)$ to a state $(T, v)$ and $c_v$ is assumed also to be independent of temperature, determine the change in the internal energy per mole, $u$.

[5pts] *e.* Determine the change in entropy per mole, $s$.

**7.** A classical system of $N$ distinguishable noninteracting particles of mass $m$ moves under the influence of a 3-dimensional harmonic potential, $U(r) = (x^2+y^2+z^2)/(2V^{2/3})$, where $V$ is the volume occupied by the particles.

[5pts] *a.* Write the canonical partition function of the system in integral form and evaluate it.

[5pts] *b.* Calculate the Helmholz free energy, $F$.

[4pts] *c.* Take $V$ as an external parameter and find the thermodynamic force, $\tilde{P}$ (conjugate to this parameter). Note: $\tilde{P} = -\big(\frac{\partial F}{\partial V}\big)_T$.

[3pts] *d.* Calculate the entropy, internal energy and the total heat capacity at constant volume.

The integrals $\int_0^\infty \mathrm{d}u\, e^{-u^2} = \sqrt{\pi}$ and $\int_0^\infty \mathrm{d}u\, u^2 e^{-u^2} = \frac{1}{4}\sqrt{\pi}$ may be useful.

# The Howard University
# Department of Physics and Astronomy

## Master of Science Comprehensive and Doctor of Philosophy Qualifying Exam

Part 2: Modern Physics

August 28, 2003

Work *four* problems, at least one from each group. Circle the numbers below to indicate your chosen problems.

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| --- | --- | --- |
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Good Luck! (And may you not need it.)

## Group A

**1.** A quantum particle of mass $m$ moves in one dimension under the influence of a potential $V(x)$ which vanishes at $x \to \pm\infty$ and has a finite minimum, but is otherwise unknown. It is in the ground state, described by $\psi_0(x) = \frac{A}{\cosh(\lambda x)}$.

[8pts] *a.* Analyzing the Schrödinger equation in the $|x| \to \infty$ limit, determine the energy of the ground state, $E_0$.

[7pts] *b.* Using the result from part *a.*, determine $V(x)$.

[4pts] *c.* Normalize $\psi_0(x)$.

[6pts] *d.* Show that for small enough oscillations about $x$=0, the system may be approximated by a linear harmonic oscillator. Determine the potential, characteristic frequency, and the energy of the ground state in this approximation, and compare the latter with the exact result in part *a.*

You may find the result $\int_0^\infty \frac{\mathrm{d}x}{\cosh(ax)} = \frac{\pi}{2a}$ useful.

**2.** Consider a quantum particle of mass $m$ moving in one dimension, under the influence of $V(x) = -\frac{\hbar^2\lambda}{2ma}\delta(x{-}a)$, where $\lambda > 0$ is a constant.

[5pts] *a.* Derive the matching conditions for $\psi(x)$ at $x$=0. (Hint: integrate the "Schrödinger equation $x \in [-\epsilon, \epsilon]$, and take the $\epsilon \to 0$ limit.)

[5pts] *b.* Determine the energies and wave-functions of the bound states.

[6pts] *c.* Now consider $V(x) = -\frac{\hbar^2\lambda}{2ma}[\delta(x{-}a){+}\delta(x{+}a)]$, and determine the ground state wave-function.

[9pts] *d.* Determine the wave-functions of all other bound states, together with any accompanying condition/restriction on $\lambda$.

**3.** Consider a 2-dimensional oscillator with $\hat{H} = \frac{1}{2m}(\hat{p}_1^2 + \hat{p}_1^2) + \frac{1}{2}m\omega^2(x_1^2 + x_1^2) + \lambda x_1 x_2$ and mass $m$ and frequency $\omega$, where the $\lambda x_1 x_2$ term provides a weak coupling ($\lambda$ is small).

[5pts] *a.* Calculate the shift in energy eigenvalues to first order in perturbation theory.

[10pts] *b.* Calculate the shift in energy eigenvalues to second order in perturbation theory.

[10pts] *c.* Calculate the energy spectrum exactly (using a canonical change of variables) and compare your exact results with the perturbative ones.

You may find $\hat{a}_i{:=}\sqrt{\frac{m\omega}{2\hbar}}(x_i + im\omega\hat{p}_i)$ and its conjugates useful in your calculations.

Group B

**4.** Consider a $P$-electron ($\ell$=1 and $s$=$\frac{1}{2}$). Use $\hat{L}_{\pm}$:=$\hat{L}_x \pm i\hat{L}_y$, and similarly for $\hat{S}_{\pm}$. For these, that the only non-zero matrix elements are: $\langle\frac{1}{2}, m'_s|\hat{S}_z|\frac{1}{2}, m_s\rangle = m_s\delta_{m'_s,m_s}$, $\langle\frac{1}{2}, m'_s|\hat{S}_{\pm}|\frac{1}{2}, m_s\rangle = \delta_{m'_s,m_s\pm1}$, $\langle 1, m'|\hat{L}_z|1, m\rangle = m\delta_{m',m}$, and $\langle 1, m'|\hat{L}_{\pm}|1, m\rangle = \sqrt{2}\delta_{m',m\pm1}$

[4pts] *a.* Show that $\hat{\vec{L}}\cdot\hat{\vec{S}} = \hat{L}_z\hat{S}_z + \frac{1}{2}(\hat{L}_+\hat{S}_- + \hat{L}_-\hat{S}_+)$.

[5pts] *b.* Construct the matrix $M = \langle 1, m'; \frac{1}{2}, m'_s|\hat{\vec{L}}\cdot\hat{\vec{S}}|1, m; \frac{1}{2}, m_s\rangle$, and show that it is a 6×6 matrix with only eight nonzero entries.

[3pts] *c.* Show that $M$ is block-diagonal, with two 1×1 and two 2×2 blocks.

[8pts] *d.* Determine the eigenvalues and eigen-kets of these matrices.

[5pts] *e.* Determine the degeneracy of each eigenvalue of $\hat{H} = \hat{H}_0 + \xi\hat{\vec{L}}\cdot\hat{\vec{S}}$, where $\xi$=*const* and $\hat{H}_0$ is the non-relativistic Hamiltonian for the H-atom including only the Coulomb interaction.

**5.** Discuss the state operator (density matrix), $\hat{\rho}$, of a spin-$\frac{1}{2}$ system.

[3pts] *a.* Explain why $\hat{\rho} = A(\mathbb{1} + \vec{a}\cdot\vec{\boldsymbol{\sigma}})$, where $\boldsymbol{\sigma}_i$ is the $i^{th}$ Pauli matrix.

[6pts] *b.* Using the normalization and Hermiticity of $\hat{\rho}$, determine $A$ and $\Im m[\vec{a}]$.

[5pts] *c.* Using the non-negativity of $\hat{\rho}$, prove that $|\vec{a}| \leq 1$.

[3pts] *d.* Write down the dynamical equation for $\hat{\rho}$. (You can also derive it, using a toy example, $\hat{\rho}_{toy} = |\psi\rangle\langle\psi|$, where $|\psi\rangle$ satisfies the usual "Schrödinger equation.)

[8pts] *e.* For $\hat{H} = \hat{H}_0\mathbb{1} - \frac{1}{2}\gamma\hbar\hat{\vec{\boldsymbol{\sigma}}}\cdot\vec{B}$, where $[\hat{\rho}, \hat{H}_0]$=0, calculate $\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t}$ and interpret your result physically.

Group C

**6.** Consider 2-state atoms in a thermal radiation field at temperature $T$. The following three processes occur:

1. $|1\rangle \to |2\rangle$ excitation through photon absorption, for which $(\frac{dN_1}{dt})_{ab} = -B_{12}N_1\rho(\nu)$;
2. $|2\rangle \to |1\rangle$ relaxation through spontaneous photon emission, for which $(\frac{dN_2}{dt})_{sp} = -A_{21}N_2$;
3. $|2\rangle \to |1\rangle$ relaxation through stimulated photon emission, for which $(\frac{dN_2}{dt})_{st} = -B_{21}N_2\rho(\nu)$.

Here $\rho(\nu) = \frac{8\pi h\nu^3}{c^3}(e^{\frac{h\nu}{kT}} - 1)^{-1}$ is the radiation density, $N_i$ is the population of the $i^{th}$ level, and $A_{ij}, B_{ij}$ are the approapriate Einstein coefficients.

[7pts] *a.* If the populations $N_1, N_2$ are in thermal equilibrium, determine $\frac{N_2}{N_1}$.

[5pts] *b.* Determine the relationship between $\frac{N_2}{N_1}$ and $e^{\frac{-h\nu}{kT}}$.

[6pts] *c.* Determine the ratio $\frac{A_{21}}{B_{21}}$ valid for very large values of $\nu$.

[7pts] *d.* Calculate the ratios $\frac{A_{21}}{B_{21}}$ and $\frac{B_{12}}{B_{21}}$ true for all values of $\nu$.

**7.** Consider a chemical reaction, $\sum_i r_i M_i = 0$, where $r_i$ molecules of type $M_i$ (and total number $N_i$) react; negative $r$'s indicate the molecules reacting to produce the ones with positive $r$'s. So, for water dissociation, $2H_2 + 1O_2 - 2H_2O = 0$, we have $r_i = 2, 1, -2$ as $i = 1, 2, 3$, and $M_i = H_2, O_2, H_2O$ are the corresponding molecule types. Write $s_j$ for a state of the $j^{th}$ molecule (regardless of its type), $E_j(s_j)$ for its energy and neglect all (non-chemical) interactions.

[5pts] *a.* Assuming all molecules *distinguishable*, show that the partition function may be written as $Z_D = \prod_i z_{i,D}$. Define $z_{i,D}$ for each type $i$; $i = 1, 2, 3$ above.

[3pts] *b.* With the molecules of the same type *indistinguishable*, show that the partition function is $Z_I = \prod_i z_{i,I}$ and explain the difference between $z_{i,D}$ and $z_{i,I}$.

[6pts] *c.* Calculate the Helmholz free energy and derive the equilibrium condition for constant temperature, $T$, total volume, $V$, and for both $Z_D$ and $Z_I$.

[6pts] *d.* Using that $dN_i = cr_i$, where $c$ is a universal constant, derive the "Law of Mass Action," $\prod_i N_i^{r_i} = K(T, V)$ from *one* of $Z_D, Z_I$; state which one, and explain why. $K(T, V)$ depends only on its arguments.

[5pts] *e.* Show that the use of the *other* partition function implies the equilibrium to exist only at $T = 0$. Consequently, are molecules of the same type distinguishable?

Stirling's approximation is: $\ln(n!) \approx n\ln(n) - n$ for large $n$. The above derivation of Gibbs's (1975) predates Quantum Mechanics, which it foreshadows, by a quarter of a century!