The Howard University
Department of Physics and Astronomy

Master of Science Comprehensive and Doctor of Philosophy Qualifying Exam

Part 1: Classical Physics

August 25, 2004

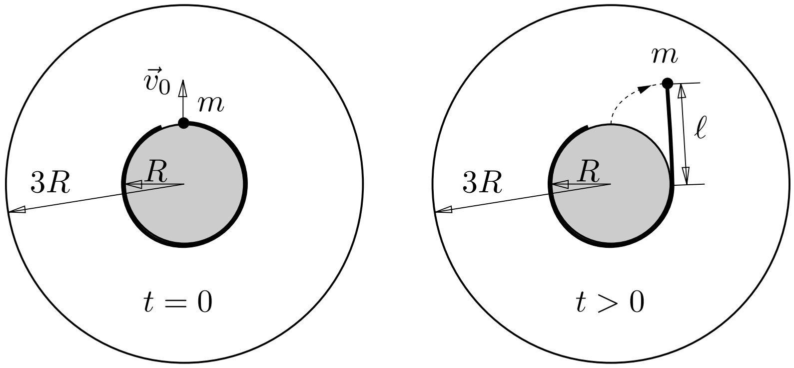
Work out the solutions to *four* problems,– at least one from each group. Circle the numbers below to indicate your choice of problems.

Group A	Group B	Group C
1 2 3	4 5	6 7

1. Write in your code-letter here: **solution**.
2. Write your code-letter and a page number (in sequential order) on the *top right-hand* corner of each submitted answer sheet.
3. Write only on one side of the answer sheets.
4. Start each problem on a new answer sheet.
5. Stack your answer sheets by problem and page number, and then staple them (at the top *left-hand corner*) with this cover sheet on the top.

Good Luck! (And may you not need it.)

1. A bead of mass m, under no external force, is attached by a massless inextensible cord which is completely wound aroud a cylinder of radius R. This cylinder is placed within a concentric cylindical shell, of radius $3R$. A radially directed kick sends the bead spiraling outward with initial velocity $\vec{v}_0$, unwinding the cord as shown:



[7pts] *a.* Using the length of the unwound piece of the cord, ℓ, as a generalized coordinate, write down the Lagrangian and determine the equation of motion.

[7pts] *b.* Find the trajectory $\ell = \ell(t)$ of the bead.

[7pts] *c.* Find the angular momentum of the bead about the axis of the cylinder and the kinetic energy after a time t.

[4pts] *d.* Find the time when the bead will hit the outer cylinder.

2. A particle of mass m moves under the influence of a potential $V(r) = Kr^4$ where $K > 0$.

[7pts] *a.* Calculate the force $\vec{F}(r)$ and make plots of both $|\vec{F}(r)|$ and $V(r)$.

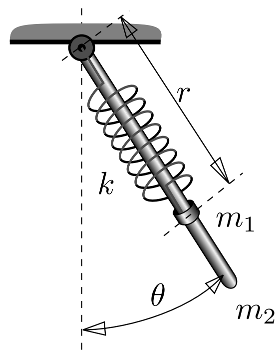
[7pts] *b.* Make a plot of the effective potential and discuss the motion of the particle without solving the equations of motion, for the cases $E < 0$, $E = 0$, and $E > 0$.

[5pts] *c.* Find the values of total energy E, the Lagrangian function L and the radius of a circular orbit.

[3pts] *d.* Calculate the period of this circular motion.

[3pts] *e.* Calculate the period of small radial oscillations, that is, the period of the motion when the particle is slightly disturbed from the circular orbit.

3. A ring with mass m_1 slides over a uniform rod which has a mass m_2 and length ℓ. The rod is pivoted at one end and hangs vertically. The ring is secured to the pivot by a massless spring with the spring constant k and unstretched length r_0, and is constrained to slide along the rod without friction. The rod and the ring are set into motion in a vertical plane. The position of the ring and the rod at time t is given by $r(t)$ and $\theta(t)$, as shown in the figure.

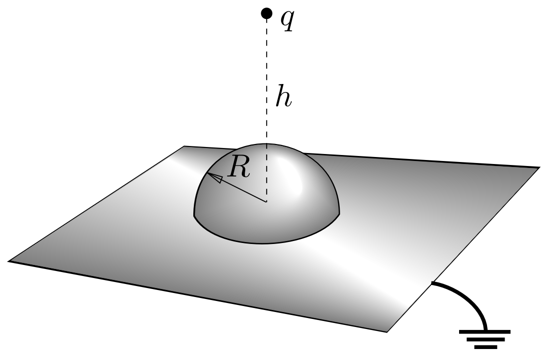


[12pts] *a.* Write the Lagrangian for the system.

[5pts] *b.* Obtain the Hamiltonian.

[8pts] *c.* Obtain the differential eqution of motion.

4. A grounded conductor has the shape of an infinite horizontal plane, with a hemispherical bulge of radius R (see the figure below). A point-charge q is placed at a distance $h > R$ above the center of the hemisphere.



[12pts] *a.* Using the method of images, determine the total electrostatic potential.

[7pts] *b.* Determine the electrostatic force on the original charge.

[6pts] *c.* Determine the lowest non-zero term in the multipole expansion of the electrostatic potential.

5. A plane electromagnetic wave is incident on the planar interface between linear, isotropic and homogeneous dielectric media of (real) indices of refraction n_1 and n_2, at an angle θ_i from the normal to the interface plane. Assume that the magnetic permeabilities of both dielectrics are $\mu_1, \mu_2 \approx \mu_0$.

[5pts] *a.* If the incident electric field, $\vec{E}_i$, is parallel to the interface, give the equalities for continuity of both the electric and the magnetic field.

[8pts] *b.* Derive the ratio of the amplitudes of the reflected and incident electric fields, $r_{TM} = |\vec{E}_{0r}|/|\vec{E}_{0i}|$, as a function of n_1, n_2 and the incident and transmitted angles, θ_i, θ_t.

[6pts] *c.* In the case when the angle between the direction of the reflected and the transmitted waves is $90°$, determine the numerical value of r_{TM}.

[6pts] *d.* In reflection from non-dielectric materials, many of the above assumptions no longer hold; in particular, consider now the case when $n_i = 1$ and $n_2 = n_R + in_I$ is complex. Determine the ratio of the *intensities* of the reflected and the incident wave, and show that for Gallium ($n_R = 3.7$ and $n_I = 5.4$), $I_r/I_i = 0.7$.

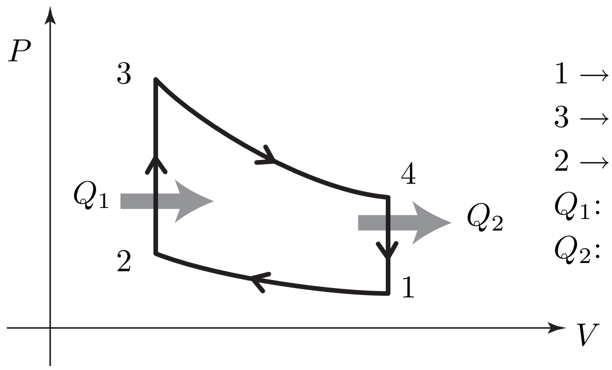
6. An ideal gas of particles, each of mass m, moving in only one dimension and at temperature T, is subject to an external force governed by the potential $V(x) = Ax^n$, where $0 \le x \le \infty$, and $A, n > 0$.

[12pts] *a.* Calculate the average potential energy per particle.

[7pts] *b.* For $n = 2$ (harmonic oscillator potential), calculate $\langle V \rangle$.

[6pts] *c.* Calculate the average potential energy per particle in a gas in a uniform gravitational field $(n = 1)$.

7. The cycle of a highly idealized gasoline engine can be approximated by the so-called Otto cycle (see figure). Treat the working medium as an ideal gas, with $\gamma \stackrel{\text{def}}{=} C_P/C_V$.



$1 \to 2$: adiabatic compression
$3 \to 4$: adiabatic expansion
$2 \to 3$ and $4 \to 1$: isohoric
Q_1: amount of heat absorbed by the gas
Q_2: amount of heat transferred from the gas

[8pts] *a.* Obtain an expression for the efficiency, η, of this cycle in terms of the compressiion ratio $r \stackrel{\text{def}}{=} V_i/V_f$

[5pts] *b.* Compute η for $\gamma = 1.4$ and $r = 10$.

[7pts] *c.* Obtain an expression for the work, W, done on the gas in the adiabatic compression process $1 \to 2$ in terms of the initial volume and pressure V_i, P_i and γ.

[5pts] *d.* Compute W for $V_i = 2\,\text{L}$ and $P_i = 1\,\text{atm}$.

The Howard University
Department of Physics and Astronomy

Master of Science Comprehensive and Doctor of Philosophy Qualifying Exam

Part 2: Modern Physics

August 27, 2004

Work *four* problems, at least one from each group. Circle the numbers below to indicate your chosen problems.

1 2 3	4 5	6 7
Group A	Group B	Group C

1. Write in your code-letter here: **solution**.
2. Place the code-letter and a page number on the *top right-hand* corner of each submitted answer sheet.
3. Write only on one side of the answer sheets.
4. Start each problem on a new answer sheet.
5. Stack your answer sheets by problem and page number, and then staple them (at the top *left-hand corner*) with this cover sheet on the top.

Good Luck! (And may you not need it.)

1. An electron is described by the wavefunction

$$\psi(x) = \begin{cases} 0 & \text{for } x < 0, \\ Ce^{-x/x_0}(1 - e^{-x/x_0}) & \text{for } x > 0, \end{cases}$$

where $x_0 = 1\,\text{nm}$ and C is a constant.

[6pts] *a.* Calculate the value of C that normalizes $\psi(x)$.

[6pts] *b.* Where is the electron most likely to be found? That is, calculate the value of x where the probability of finding the electron is the largest.

[8pts] *c.* Calculate the expectation value, $\langle x \rangle$, for this electron and compare your results with the most likely position. Comment on any differences.

[5pts] *d.* Calculate the indeterminacy, $\triangle x$.

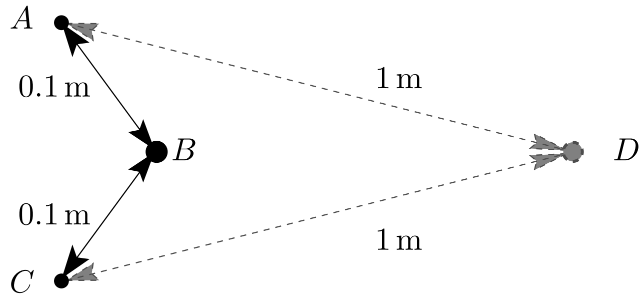
2. Consider a relativistic particle of rest mass m, momentum p, and total energy E.

[8pts] *a.* Show that $E^2 = p^2 + m^2c^4$.

[8pts] *b.* Show that the "length" of the energy-momentum four-vector of a particle is invariant with respect to all Lorentz transformations.

[9pts] *c.* A proton-proton collision can create a π^0 meson as an additional particle in the final state, if there is sufficient energy. The observed reaction is $p + p \to p + p + \pi^0$. If the initial state consists of a proton of kinetic energy K colliding with a proton at rest, calculate the minimum value of K for which the reaction may occur.

3. Charges of $5\,\mu$C are located at points A and C, as shown below.



[10pts] *a.* A bead of 15 g mass and $5\,\mu$C charge is released from rest at point B; calculate its speed at point D.

[15pts] *b.* Redo this calculation for a bead of 40×10^{-15} g mass; neglect radiative effects.

(Remark: The distance between points A and C is immaterial for answering the questions!)

4. An operator Q satisfies the relations

$$\big[\,[\,Q\,,\,\vec{J}^2\,]\,,\,\vec{J}^2\,\big] = \tfrac{1}{2}\big(Q\vec{J}^2 + \vec{J}^2 Q\big) + \tfrac{3}{16}Q\ , \qquad [\,Q\,,\,J_z\,] = m_q Q\ ,$$

where $\vec{J}$ is the usual (total) angular momentum (vector) operator and J_z the component in the z direction.

[6pts] *a.* For the matrix element $\langle j', m' |\, Q \,| j, m\rangle$ to be non-zero, use the first relation to determine the allowed values $\triangle j = j' - j$.

[6pts] *b.* For the matrix element $\langle j', m' |\, Q \,| j, m\rangle$ to be non-zero, use the second relation to determine the allowed values $\triangle m = m' - m$ in terms of m_q.

[3pts] *c.* Given your results for a. and b., what are the two possible values for m_q?

[5pts] *d.* Calculate $\langle j', m' |\, [\,Q\,,\,\vec{J}^2\,] \,| j, m\rangle$ in terms of $\langle j', m' |\, Q \,| j, m\rangle$, j and $\triangle j$.

[5pts] *e.* Writing Q and $\bar{Q}$ for the two operators corresponding to the two possible values of m_q, prove that $Q\bar{Q}$ and $\bar{Q}Q$ commute with J_z.

Hint: "Sandwich" the given relations between $\langle j', m'|$ and $|j, m\rangle$.

5. Consider a 2-dimensional harmonic oscillator, for which the Hamiltonian can be written as $H = \frac{1}{2m}({p_x}^2 + {p_y}^2) + \frac{1}{2}m\omega^2(x^2 + y^2)$.

[4pts] *a.* Write down energies of the allowed states of this oscillator (in units of $\hbar\omega$) and specify their degeneracy.

[4pts] *b.* For a suitably small constant α, does a perturbation of the form $V = \alpha x$ change the degeneracy? Why (why not)?

[4pts] *c.* For a suitably small constant β, does a perturbation of the form $V = \beta x^2$ change the degeneracy? Why (why not)?

[4pts] *d.* For a suitably small constant γ, does a perturbation of the form $V = \gamma x^4$ change the degeneracy? Why (why not)?

[4pts] *e.* Use perturbation theory to calculate the first order shift in the ground state energy, caused by a small perturbation $V = \gamma x^4$.

[5pts] *f.* For all of the above perturbations and for any arbitrary collection of states, is it necessary to use *degenerate* perturbation theory? Why (why not)?

6. Consider a particle of mass M constrained to move on a circle of radius a in the x, y-plane.

[5pts] *a.* Write down the Schrödinger equation in terms of the usual cylindrical-polar angle ϕ.

[5pts] *b.* Determine the complete set of states, the corresponding energy spectrum and orthonormalize the stationary states.

[5pts] *c.* Assume now that the particle has charge q and is placed in a small electric field $\vec{E} = \mathcal{E}\hat{e}_x$. Determine the first non-zero perturbative correction to the energy levels.

[5pts] *d.* Instead of the electric field, apply a small magnetic field $\vec{B} = \mathcal{B}\hat{e}_z$. Determine the first non-zero perturbative correction to the energy levels.

[5pts] *e.* What is the degeneracy of the unperturbed system (the one with $\mathcal{E} = 0 = \mathcal{B}$)? And with $\mathcal{E} \neq 0 = \mathcal{B}$? And with $\mathcal{E} = 0 \neq \mathcal{B}$?

7. Consider an $L \times L \times L$ cube of metal, wherein the electrons may be treated as if comprised of an ideal gas confined in the cube.

[5pts] *a.* Write down the wave-function for the electron states and the expression for the energy levels, $E_{\vec{n}}$, where $\vec{n} = (n_x, n_y, n_z)$.

[5pts] *b.* Let $n = |\vec{n}|$. Determine the number of state between n and $n + \mathrm{d}n$. Using the relation between $E_{\vec{n}}$ and $\vec{n}$, and $\mathrm{d}E_{\vec{n}}$ and $\mathrm{d}n$, eliminate n and $\mathrm{d}n$ and obtain the number of states, $\mathrm{d}N$, within $[E, E + \mathrm{d}E]$.

[5pts] *c.* Determine the Fermi energy, that is, the energy of the highest occupied state.

[5pts] *d.* Determine the average kinetic energy of these electrons.

[5pts] *e.* Determine the pressure of this ideal gas of electrons.