The Howard University
Department of Physics and Astronomy

Master of Science Comprehensive and Doctor of Philosophy Qualifying Exam

Part 1: Classical Physics

August 29, 2005

Work out the solutions to *four* problems,– at least one from each group. Circle the numbers below to indicate your choice of problems.

Group A	Group B	Group C
1 2 3	4 5	6 7

1. Write in your code-letter here: ☐ .
2. Write your code-letter and a page number (in sequential order) on the *top right-hand* corner of each submitted answer sheet.
3. Write only on one side of the answer sheets.
4. Start each problem on a new answer sheet.
5. Stack your answer sheets by problem and page number, and then staple them (at the top *left-hand corner*) with this cover sheet on the top.

Good Luck! (And may you not need it.)

Group A

1. The system of a particle of mass m acted on by an inverse square force, with potential $V(r) = +k/r$, has a Lagrangian

$$L = \frac{m}{2}\left[\,\dot{r}^2 + r^2\dot{\theta}^2\,\right] - \frac{k}{r}$$

in polar coordinates (r, θ).

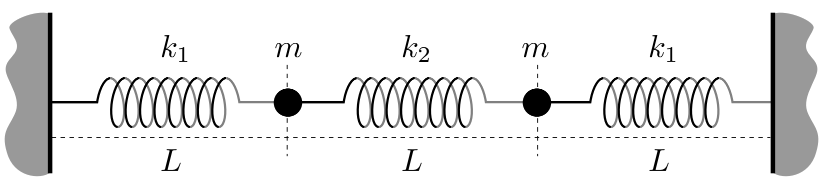
[7 pts] *a.* Write down the expressions for the canonical momenta conjugate to r, θ.

[7 pts] *b.* Construct the Hamiltonian, H, of this system.

[7 pts] *c.* Obtain Hamilton's equations of motion.

[4 pts] *d.* Is H a constant of motion? Why or why not? Explain.

2. Examine two equal masses attached to springs and constrained to move in one (horizontal) dimension, as shown below:



[5 pts] *a.* Determine the frequencies of the normal modes in this system.

[7 pts] *b.* Determine and sketch the normal modes in this system.

[7 pts] *c.* Obtain the normalized eigenvectors for the motion of the masses.

[6 pts] *d.* Next, the left mass is displaced by $L/2$ towards the right one, which is held stationary at its equilibrium position. Both masses are then released from rest at $t = 0$. Determine the subsequent motion of the masses.

3. The science-fiction writer R.A. Heinlein describes the "skyhook" geostationary satellite to consist, in its simplest form, of a cable of uniform mass per length (λ), placed in the equatorial plane, oriented radially and extending from just above the surface of the planet to a length L above the planet.

[6 pts] *a.* Write down the condition for "skyhook" to be in equilibrium by considering an infinitesimal segment of the cable.

[7 pts] *b.* Find the required length of "skyhook" for it to orbit around the Earth.

[7 pts] *c.* Determine the height at which a conventional geostationary satellite has to be positioned, and compare that with the length of the "skyhook".

[5 pts] *d.* Consider finally a conventional geostationary satellite (in the equatorial plane) of mass m which supports a cable of mass per unit length λ, hanging down to the Earth surface. Find the equation determining the height at which such a satellite has to orbit and show that it depends only on the ratio m/λ, not on m and λ separately. (You need not solve this equation.)

The radius of Earth is $R_E = 6.4 \times 10^6$ m; the gravitational acceleration at the surface is $9.81\,\mathrm{m/s^2}$.

Group B

4. Examine a dielectric sphere, of radius a and dielectric constant ϵ_1, embedded in another dielectric medium (ϵ_2) with an asymptotically homogeneous electric field E_0 oriented parallel to the z-axis.

[7 pts] *a.* Determine the potential throughout all space.

[5 pts] *b.* Determine the bound surface charge density at $r = a$.

[6 pts] *c.* Suppose now that a long wavelength electromagnetic wave of amplitude E_0 and frequency ω polarized in the $\hat{z}$ direction is incident on the sphere traveling parallel to the x-axis. Demonstrate the form of the incident wave.

[7 pts] *d.* Determine the asymptotic form of the electromagnetic radiation generated by the dielectric sphere. It is sufficient to determine the form of the scalar and vector potentials.

5. A plane electromagnetic wave of frequency ω and wavenumber k propagates in the positive z direction. For $z < 0$, the medium is air and the conductivity is $\sigma_a = 0$. For $z > 0$, the medium is a lossy dielectric, with dielectric constant κ and $\sigma_d > 0$. Assume that both air and dielectric are nonmagnetic, and that $\vec{E}, \vec{B}$ are in the (x, y)-plane.

[10 pts] *a.* Show that the dispersion relation (relation between k and ω) in the lossy medium is

$$k^2 = \frac{\omega^2}{c^2}\big(\kappa + i\frac{4\omega\sigma_d}{\omega}\big) \ .$$

[5 pts] *b.* Find the values of η and ξ if k is written as

$$k = \frac{\omega}{c}\big(\eta + i\xi\big) \ .$$

[5 pts] *c.* Find the limiting value of k for a very poor conductor ($\sigma_d \ll \kappa\omega$), and for a very good conductor ($\sigma_d \gg \kappa\omega$).

[5 pts] *d.* Find the e^{-1} penetration depth δ (“skin depth”), for the plane wave power in the case $\sigma_d \gg \kappa\omega$.

Group C

6. Consider a system of $N \gg 1$ non-interacting particles in which the energy of each particle can assume two and only two disticnt values: 0 and E ($E > 0$). Denote by n_0 and n_1 the occupation numbers of the energy levels o and E, respectively. Let the fixed total energy of the system be U.

[3 pts] *a.* Write down the total number of available states, Ω.

[6 pts] *b.* Use Stirling's approximation to obtain the entropy of the system, $S = S(N, n_0, n_1)$.

[2 pts] *c.* Write down expressions for the total number, N, of particles and the total energy, U, of this system.

[7 pts] *d.* For a constant number of particles, determine the temperature, $T = T(U, E, N)$.

[7 pts] *e.* Determine the range of values of n_0 so that $T < 0$.

7. An ideal gas undergoes adiabatic expansion.

[5 pts] *a.* Obtain the temperature as a function of pressure.

[5 pts] *b.* Impose the condition of gravitational equilibrium at a height, h, above the ground and obtain a differential relationship between $\mathrm{d}P$ and $\mathrm{d}h$ from the ideal gas law. (Assume h to be much smaller that Earth's radius.)

[5 pts] *c.* A simple model of the atmosphere assumes that the temperature is independent of h. Find P as a function of h for this simple model (Boltzmann's relation).

[5 pts] *d.* A more realistic model neglects thermal conduction and assumes that as the mass of the air rises or descends, it undergoes and adiabatic reversible process. Obtain an expression for $\frac{\mathrm{d}T}{\mathrm{d}h}$ under this assumption.

[5 pts] *e.* Given $C_P/C_V = 1.4$ and $m = 0.029\,kg/\text{mole}$ for air, and $k = 1.381 \times 10^{-23} J/K$, $N_A = 6.022 \times 10^{23}$, $g = 9.80\,m/s$, obtain an estimate for the change of the temperature with altitude at $h = 0$, and for the change in pressure with altitude at $P = P_0 = 0$ and $h = 0$.

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Part 2: Modern Physics

August 31, 2005

Work *four* problems, at least one from each group. Circle the numbers below to indicate your chosen problems.

1 2 3	4 5	6 7
Group A	Group B	Group C

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Good Luck! (And may you not need it.)

Group A

1. A particle has the wave-function:

$$\psi(r,\theta,\phi) = C\, f(r)\, \sin^2\theta\,\big(2-\sqrt{7}\cos\theta\big)\,\sin(2\phi)\ ,$$

where $f(r)$ is a normalized radial function.

[5 pts] *a.* Express the angular part of this wave-function as a superposition of the spherical harmonics $Y_\ell^m(\theta,\phi)$.

[3 pts] *b.* Determine $|C|$ so that $\int_0^\infty r^2\mathrm{d}r\,|f|^2 = 1$ is maintained.

[7 pts] *c.* Calculate the probabilities that a measurement of $\vec{L}^2$ and L_z will yield the values given in the table (fill in all blanks):

Operator	**Measurements and Probabilities**						
$\vec{L}^2$ **:**	0	2	4	6	9	12	16
Prob. =							
L_z **:**	−3	−2	−1	0	+1	+2	+3
Prob. =							

[5 pts] *d.* Calculate the expectation values $\langle\psi|\vec{L}^2|\psi\rangle$ and $\langle\psi|L_z|\psi\rangle$.

[5 pts] *e.* Calculate the rms uncertainties in $\vec{L}^2$ and L_z.

Some spherical harmonics:

$$Y_0^0=\sqrt{\tfrac{1}{4\pi}}\ ,\quad Y_1^0=\sqrt{\tfrac{3}{4\pi}}\cos\theta\ ,\quad Y_2^0=\sqrt{\tfrac{5}{16\pi}}(3\cos^2\theta-1)\ ,\quad Y_3^0=\sqrt{\tfrac{7}{16\pi}}(5\cos^3\theta-3\cos\theta)\ ,$$

$$Y_1^1=-\sqrt{\tfrac{3}{8\pi}}\sin\theta\,e^{i\phi}\ ,\quad Y_2^1=-\sqrt{\tfrac{15}{8\pi}}\sin\theta\,\cos\theta\,e^{i\phi}\ ,\quad Y_3^1=-\sqrt{\tfrac{21}{64\pi}}(5\cos^2\theta-1)\sin\theta\,e^{i\phi}\ ,$$

$$Y_2^2=\sqrt{\tfrac{15}{32\pi}}\sin^2\theta\,e^{2i\phi}\ ,\quad Y_3^2=\sqrt{\tfrac{105}{32\pi}}\sin^2\theta\,\cos\theta\,e^{2i\phi}\ ,\quad Y_3^3=-\sqrt{\tfrac{35}{64\pi}}\sin^3\theta\,e^{3i\phi}\ .$$

2. Examine the pionic decays of K^0 governed by weak interactions. The dominant decays are $K^0 \to \pi^+\pi^-$ and $K^0 \to \pi^0\pi^0$. Total isospin is *not* conserved in these processes, but changes either by $\triangle I = +\frac{1}{2}$ or by $\triangle I = -\frac{1}{2}$.

[5 pts] *a.* Write down $|I, m_I\rangle$ states for each of $K^0, \bar{K}^0, \pi^\pm, \pi^0$, based on the data given below. Specify the smallest consistent I^2 and I_z eigenvalues for the Hamiltonian terms, $\hat{H}_{+-}, \hat{H}_{00}$, mediating respectively the two stated decays.

[5 pts] *b.* Using the basic particle data (below), determine the possible values of the angular momentum and isospin for each given two-pion system into which the K^0 decays.

[5 pts] *c.* Based only on the particle data below, estimate the ratio of cross-sections $\frac{\sigma(K^0 \to \pi^+\pi^-)}{\sigma(K^0 \to \pi^0\pi^0)}$.

[5 pts] *d.* If the charge-conjugation & parity (reflection) operator acts as $\mathrm{CP}\big|K^0\big\rangle = |\bar{K}^0\rangle$ and obeys $(\mathrm{CP})^2 = \mathbb{1}$, find the ortho-normalized kaon (K) CP-eigenstates.

[5 pts] *e.* For $|K_\pm\rangle$, such that $\mathrm{CP}|K_\pm\rangle = \pm|K_\pm\rangle$, let $\Gamma_\pm$ denote the decay rate. What is the fraction of K^0's in an initially pure K^0-beam, as a function of proper time, τ?

[5 pts] *f.* With no new terms in the Hamiltonian, is the decay $K^0 \to \pi^+\pi^0\pi^-$ possible? Is the decay $K^0 \to \pi^+\pi^0\pi^0\pi^-$ possible? Prove your assertions by a short calculation.

The $K^0, \pi^\pm, \pi^0$ mesons have no spin and are odd under parity (space reflection). Use that $I_z(\bar{K}^0) = -I_z(K^0) = +\frac{1}{2}$, while the $I_z(\pi^\pm) = \pm 1$ and $I_z(\pi^0) = 0$. The rest-masses are (in MeV/c^2): $m_{K^0} = 497.7$, $m_{\pi^0} = 135.0$, $m_{\pi^\pm} = 139.6$. Some possibly useful Clebsh-Gordan coefficients: $\langle 1,1;1,-1|2,0\rangle = 1/\sqrt{6}$, $\langle 1,1;1,-1|1,0\rangle = 1/\sqrt{2}$, $\langle 1,1;1,-1|0,0\rangle = 1/\sqrt{3}$, $\langle 1,1;0,0|2,0\rangle = \sqrt{2/3}$, $\langle 1,1;0,0|1,0\rangle = 0$, $\langle 1,1;0,0|0,0\rangle = -1/\sqrt{3}$.

3. The atomic number of the sodium (`Na`) atom is $Z = 11$.

[7 pts] *a.* Write down the electronic configuration for the ground state of the `Na` atom, showing in standard $(s, p, d, f, \ldots)$ notation the assignment of all the electrons to various one-electron states. Give the standard spectroscopic notation for the ground state of the `Na` atom in the form $n^{2S+1}L_J$, where n is the principal quantum number.

[8 pts] *b.* The lowest frequency line in the absorption spectrum of the `Na` atom gives rise to a doublet. These two lines are called the D lines of sodium and occur at wavelengths of 588.9950 and 588.5924 nm, respectively. What is the mechanism responsible for the two D lines in the sodium optical spectrum? Draw an energy-level diagram showing the radiative transitions responsible for these lines.

[10 pts] *c.* In the sodium atom the single valence electron sees the core as a spherically symmetric distribution of charge. Every sodium energy-level for nonzero angular momentum has a fine structure associated with it due to spin-orbit interaction of the valence electron. Using central-field model, write down the spin-orbit Hamiltonian, assuming a Coulomb potential $U(r) = -\frac{1}{4\pi\epsilon_0}\frac{Z'e^2}{r}$. Determine explicitly the effective Z', the spin-orbit perturbation energy and show that it is proportional to n^{-3}.

Group B

4. Consider a simple quantum particle moving in a one-dimension and under the influence of a restoring force $F = -kx^5$, $k > 0$.

[5 pts] *a.* Write down the Schrödinger equation for the wave-function $\Psi(x)$ for the stationary states of such a particle.

[5 pts] *b.* Show that for very large x, $\Psi(x)$ must decay exponentially with some power of x. (Why?) Find this asymptotic wave-function $\Psi_{\rm asymp}(x) = \phi(x)$.

[5 pts] *c.* Writing $\Psi(x) = \phi(x)f(x)$, find the differential equation that determines $f(x)$.

[5 pts] *d.* Write down the wave-functions $\Psi(x)$ in the WKB approximation, and the integral condition that determines the bound-state energies of these states. You need not solve this integral.

[5 pts] *e.* Consider now varying k and *independently* n for a restoring force $F = -kx^n$. Explain which of these two variations is (is not) *adiabatic* (= preserves the qualitative features of the energy spectrum), and why (why not).

5. A non-relativistic quantum particle moves in one dimension, denoted x, under the influence of an *arbitrary* real potential $V(x)$, bounded by $V(x) \geq V_0$, $|x| < \infty$.

[3 pts] *a.* Write down the Hamiltonian and state the canonical commutation relation, $[\hat{x}, \hat{p}] = ?$

[4 pts] *b.* Evaluate $[\hat{H}, \hat{x}]$ and $\big[\hat{x}, [\hat{H}, \hat{x}]\big]$.

[3 pts] *c.* Calculate the expectation value of $\big[\hat{x}, [\hat{H}, \hat{x}]\big]$ in the ground state, $|0\rangle$.

[5 pts] *d.* By expanding the double commutator $\big[\hat{x}, [\hat{H}, \hat{x}]\big]$ and inserting complete sets of intermediate states, $|n\rangle$, obtain the Thomas-Reiche-Kuhn sum rule:

$$\frac{2m}{\hbar^2} \sum_{n=0}^{\infty} (E_n - E_0) \big|\langle n|\hat{x}|0\rangle\big|^2 = 1 \ .$$

[5 pts] *e.* By expanding the triple commutator $[\hat{x}, [\hat{x}, [\hat{H}, \hat{x}]]]$ and inserting complete sets of intermediate states prove:

$$\sum_{k,n=0}^{\infty} (E_k - E_n) \langle 0|\hat{x}|k\rangle \langle k|\hat{x}|n\rangle \langle n|\hat{x}|0\rangle = 0 \ .$$

[5 pts] *f.* For a 3-dimensional system with coordinates $\hat{x}_i$, $i = 1, 2, 3$, derive the generalization:

$$\sum_{k,l,n=0}^{\infty} \frac{2m}{\hbar^2} (E_{k,l,n} - E_{0,0,0})\, \Re e \big[\langle 0,0,0|\hat{x}_i|k,l,n\rangle \langle k,l,n|\hat{x}_j|0,0,0\rangle\big] = \delta_{ij} \ ,$$

where $\Re e[z] \stackrel{\rm def}{=} \frac{1}{2}(z + z^*)$ denotes the real part of z, $\forall z$.

Do not assume any of the facts characteristic of the linear harmonic oscillator only!

Group C

6. Consider a particle of mass M constrained to move on a circle of radius a in the x, y-plane.

[4 pts] *a.* Write down the Schrödinger equation in terms of the usual cylindrical-polar angle ϕ.

[5 pts] *b.* Determine the complete set of states, the corresponding energy spectrum and orthonormalize the stationary states.

[5 pts] *c.* Assume now that the particle has charge q and is placed in a small electric field $\vec{E} = \mathcal{E}\hat{e}_x$. Determine the first non-zero perturbative correction to the energy levels.

[5 pts] *d.* Instead of the electric field, apply a small magnetic field $\vec{B} = \mathcal{B}\hat{e}_z$. Determine the first non-zero perturbative correction to the energy levels.

[6 pts] *e.* Determine the degeneracy of the unperturbed system (the one with $\mathcal{E} = 0 = \mathcal{B}$). Similarly, with $\mathcal{E} \neq 0 = \mathcal{B}$, and then with $\mathcal{E} = 0 \neq \mathcal{B}$.

7. The interaction Hamiltonian for a spin-$\frac{1}{2}$ particle in a magnetic field is given by $H = \vec{\omega}\cdot\hat{\vec{S}}$, where the matrix representation of the spin operators is given by $[\hat{\vec{S}}] = \frac{1}{2}\hbar\,\vec{\boldsymbol{\sigma}}$, in terms of the usual Pauli matrices $\boldsymbol{\sigma}^i$, $i = 1, 2, 3$.

[5 pts] *a.* Show that the time evolution operator for the quantum state vectors takes the form $\boldsymbol{U}(t) = \exp\{i\,\mathbb{M}\,t\}$, where $\mathbb{M}^2 = \Omega^2\,\mathbb{1}$. Determine Ω in terms of $\vec{\omega}$.

[4 pts] *b.* Expand the time evolution matrix to show that it is proportional to a linear combination of only $\mathbb{1}$ and $\mathbb{M}$; determine the coefficients in that linear combination.

[4 pts] *c.* Determine $\mathbb{M}$ in terms of $\mathbb{1}, \boldsymbol{\sigma}^i$, and determine the eigenvalues of $\mathbb{M}$.

[6 pts] *d.* If the system is in the state $\left|m_s = +\frac{1}{2}\right\rangle$ at time $t = 0$, determine the state of the system at a later time, $t > 0$.

[6 pts] *e.* Determine the probability that the system is observed to be in state $\left|m_s = +\frac{1}{2}\right\rangle$ at a later time, $t > 0$.

In case you forgot: $\boldsymbol{\sigma}^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\boldsymbol{\sigma}^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\boldsymbol{\sigma}^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and they satisfy the relations $[\boldsymbol{\sigma}^j, \boldsymbol{\sigma}^k] = 2i\epsilon^{jkl}\boldsymbol{\sigma}^l$ and $\{\boldsymbol{\sigma}^j, \boldsymbol{\sigma}^k\} = 2\delta^{jk}\,\mathbb{1}$, where $[A, B] \stackrel{\text{def}}{=} AB - BA$ and $\{A, B\} \stackrel{\text{def}}{=} AB + BA$.