The Howard University
Department of Physics and Astronomy

Master of Science Comprehensive and Doctor of Philosophy Qualifying Exam

Part 1: *Classical Physics*
August 28, 2007

Work *four* problems, at least one from each group. Circle the numbers below to indicate your chosen problems.

1	2 3	4 5 6
Group A	Group B	Group C

1. Write in your code letter here: ☐
2. Write your code letter and a page number (in sequential order) on the *top right-hand* corner of each submitted answer sheet.
3. Write only on one side of each answer sheet.
4. Start each problem on a new answer sheet.
5. Stack your answer sheets by problem and page number, and then staple them (at the top *left-hand corner*) with this cover sheet on the top.

1.

☐ Write down Maxwell's equations for a medium with dielectric function ϵ, permeability μ, and conductivity σ.

[5pts] a. Show that there is a plane wave solution with frequency ω, and obtain an expression for the complex propagation vector $\vec{k}$.

[5pts] b. Obtain an expression for the the decay constant (skin depth) of the plane wave if this medium is a good conductor at the plane wave frequency ($\sigma \geq \omega\epsilon/4\pi$).

[5pts] c. A simple model for the conductivity of this medium can be obtained by solving the equation of motion for electrons (mass m, charge e, velocity $\vec{v}$, collision frequency γ) in a field $\vec{E}_0$, in this medium:

$$m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + mg\vec{v} = e\vec{E}_0 e^{i\omega t}$$

and use $\vec{j} = en\vec{v} = \sigma\vec{E}$, where n is the electron density. Using this expression, calculate the skin depth in a tenuous plasma (γ=0) in terms of ω_p^2=$(4\pi ne^2)/m$.

[5pts] d. In the ionosphere (a tenuous plasma), an external static magnetic induction, $\vec{B}_0$ is present, and neglecting transverse fields, the magnetic field is in the z-direction. Consider the plane wave to be circularly polarized and propagating in the z-direction also, so that E=$E_0(\hat{e}_1 + i\hat{e}_2)$. The equation of motion for the electrons in the ionosphere becomes

$$m\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} \approx e\vec{E}_\pm e^{-i\omega t} + \frac{e}{c}\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\times\vec{B}_0 \ .$$

Solve this for the displacement $\vec{x}$ and hence the polarization, to show that the index of refraction is different for left- and right-circularly polarized radiation.

[5pts] e. Find the frequency at which only one circular polarization component will propagate without attenuation, in terms of ω_p and ω_B=$\frac{eB_0}{mc}$

2. A particle of mass *m* moves under the action of a force the potential of which is given by $V(r) = kr^4$ where $k > 0$.

a. Calculate the force f(r) and plot V(r) *vs*. r and f(r) *vs*. r. [5 pts]

b. Plot $V_{eff}(r)$ *vs*. r. Discuss the motion of the particle for energies: $E=V_0$, $E > V_0$ and $E < V_0$, where $V_0 = V_{eff}(r_0)$ is the minimal value of $V_{eff}(r)$. (Recall: V_{eff} includes, in addition to V(r), also the "centrifugal potential" term.) [7 pts]

c. Find the values of the angular momentum l and energy E for the radius $r=r_0$ of a circular orbit. [6 pts]

d. Calculate the period T of circular motion with radius r_0. [7 pts]

3.

Given are the Maxwell equations in Gaussian units:

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\,, \qquad \vec{\nabla}\cdot\vec{D} = 0\,,$$

$$\vec{\nabla}\cdot\vec{B} = 0\,, \qquad \vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}\,;$$

a vector identity $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$, and that $c = 3\times10^{10}$ cm/s.

It is desired to calculate the reflectivity, R, from a metal surface at normal incidence. Same values of ε and μ may be used for vacuum and metal. However, there is a large conductivity σ associated with the metal.

[10pts] a. What are the wave equations for a plane wave of frequency ω in the vacuum and the metal? What are the solutions of these equations?

[10pts] b. How do you match the boundary conditions at the metal surface?

[5pts] c. How do you calculate R? For visible light, $\omega = 3\times10^{15}\,\mathrm{s}^{-1}$. Estimate R for copper for which $\sigma = 5\times10^{17}\mathrm{statohm}^{-1}\mathrm{cm}^{-1} = 6\times10^{7}\mathrm{ohm}^{-1}\mathrm{m}^{-1}$.

4. Consider a single (nonrelativistic) particle of mass m and charge q moving in an electromagnetic field.

a. Write down the Lagrangian for the system in Cartesian position coordinates. [5 pts]
b. Write down the Hamiltonian for the system [5 pts]
c. Write down the canonical momenta for the system [5pts]
d. Write down Hamilton's equations using the Hamiltonian obtained from the second step [5 pts]
e. Using Hamilton's equations, obtain an expression for the Lorentz force law. [5 pts]

5. Write down expressions for the following.

a. An expression for the symplectic condition for a canonical transformation if M is the symplectic matrix (or Jacobian matrix of the transformation).[10 pts]

b. Obtain the symplectic matrix for the following transformation [10 pts]

$$Q = q \cos a - p \sin a$$

$$P = q \sin a + p \cos a$$

c. Is the transformation of the previous step canonical? Explain. [5 pts]

6. Consider a one-dimensional harmonic oscillator of mass m and spring constant k.

a. Write the Hamiltonian in terms of the canonical coordinate q and the conjugate momentum p. [7 pts]
b. Using Poisson bracket $\{Q_i,P_j\}=\delta_{ij}$, find the condition necessary for a change of variables $(q,p) \rightarrow (Q,P)$ to be a canonical transformation. [6 pts]
c. Using the results from part b., determine the value of the constant C so that the equations

$$Q = C(p + im\omega q) \quad \text{and} \quad P = C(p - im\omega q), \quad \omega = \sqrt{k/m},$$

define a canonical transformation. [5 pts]
d. Determine the generating function $S(q,P)$ for the canonical transformation in part c. [7 pts]

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Part 2: *Modern Physics*
August 30, 2007

Work *four* problems, at least one from each group. Circle the numbers below to indicate your chosen problems.

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1. Consider a two level system with eigenstates $|1\rangle$ and $|2\rangle$ having, respectively, energies ε_1 and ε_2, and $\varepsilon_2 > \varepsilon_1$.

a. If the system is in the state $|\psi\rangle = a|1\rangle + b|2\rangle$, what is the density matrix ρ ? [15 pts]
b. Let the system be in thermal equilibrium at temperature T. What is ρ ? What is F? What is S? [10 pts]

2. Consider a monatomic crystal consisting of N atoms. These may be situated in two kinds of positions:

a. Normal position, indicated by 0
b. Interstitial position, indicated by X

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0       0       0
    X       X
0       0       0
    X       X
0       0       0
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Suppose that there is an equal number (=N) of both kinds of position, but that the energy of an atom at an interstitial position is larger by an amount ε than of an atom at a normal position. At T=0 all atoms will therefore be in normal position. Show that, at a temperature T, the number n of atoms at interstitial sites is

$$n = Ne^{-\varepsilon/2kT} \text{ for } n \ll N$$

Use the fact that the Helmholtz free energy is a minimum for equilibrium at constant volume and temperature

3.

Consider the linear harmonic oscillator (LHO) with characteristic frequency ω. With $\hat{x}$ and $\hat{p}$ denoting its position and linear momentum operators, respectively, we define $\hat{a} := \sqrt{\frac{m\omega}{2\hbar}}\big(\hat{x} + \frac{i}{m\omega}\hat{p}\big)$ and denote its Hermitian conjugate by $\hat{a}^\dagger$. Use the standard basis, $\mathscr{H} = \{\,|n\rangle\,,\ n = 0, 1, 2, \cdots\}$, where $\hat{a}\,|n\rangle = \sqrt{n}\,|n{-}1\rangle$ and $\hat{a}^\dagger\,|n\rangle = \sqrt{n{+}1}\,|n{+}1\rangle$.

a. Derive the commutation relation $[\,\hat{a}\,,\,\hat{a}^\dagger\,]$ from the canonical relation, $[\,\hat{x}\,,\,\hat{p}\,] = i\hbar$.

b. Construct an eigenstate, $|z\rangle$, of the annihilation operator: $\hat{a}\,|z\rangle := z\,|z\rangle$ as a formal expansion $|z\rangle := \sum_n c_n\,|n\rangle$. From the defining condition of such a state, determine as many of the constants c_n as you can, in terms of the eigenvalue, z.

c. Re-express your result as $|z\rangle = \hat{Z}_z\,|0\rangle$, determine the correctly normalized operator $\hat{Z}_z$, and all the values of the eigenvalue z permitted by normalizability of $|z\rangle$.

d. Calculate $\hat{U}^{-1}(t)\,\hat{a}\,\hat{U}(t)$, where $\hat{U}(t) := e^{-i\frac{t}{\hbar}\hat{H}}$ is the evolution operator.

e. Calculate $\hat{U}(t)\,|z\rangle$. Determine from this a physical interpretation of z, and of $|z\rangle$.

Be carefully with the limits on summations over the $|n\rangle$. You may need the Baker-Campbell-Hausdorff formulae, $e^A e^B = e^B e^A e^{[A,B]}$ and $e^A e^B = e^{A+B-\frac{1}{2}[A,B]}$, both valid if $[A,[A,B]] = 0 = [B,[A,B]]$, and the generally valid

$$e^{\alpha A}\,B\,e^{-\alpha A} = B + \sum_{k=1}^{\infty}\frac{\alpha^k}{k!}\Big[\underbrace{A,[\,\cdots,[\,A}_{k},\,B\,]\cdots\Big]\,.$$

4. The quantum states available to a given physical system are (i) a group of g_1 equally likely states with a common energy value $\square_1$ and (ii) a group of g_2 equally likely states, with a common energy value $\square_2$. Show that the entropy of the system is given by

$$S = -k\left[p_1 \ln(p_1 / g_1) + p_2 \ln(p_2 / g_2)\right]$$

where p_1 and p_2 are, respectively, the probabilities of the system being in a state belonging to group 1 or to group 2: $p_1 + p_2 = 1$.

a. a. Assuming that the p's are given by a canonical distribution, show that

$$S = k\left[\ln g_1 + \ln\{1 + (g_2 / g_1)e^{-x}\} + \frac{x}{1 + (g_1 / g_2)e^{x}}\right]$$

where $x = (\square_2 - \square_1)/kT$, assumed positive. [10 pts]

b. Verify the foregoing expression for S by deriving it from the partition function of the system. [10 pts]

c. Check that as $T\square 0$, $S\square k \ln g_1$. Interpret this result physically. [5 pts]

5.

Consider a heteronuclear diatomic molecule with the moment of inertia I, which is constrained to rotate in a plane.

a. Show that the energy levels are given by $E_j = \frac{\hbar^2}{2I} j(j+1)$.

b. Determine the allowed values of j, and the degeneracy of the corresponding energy levels.

c. Using quantum statistical mechanics, find expressions for the partition function Z and the average energy $\langle E \rangle$ of this system. (You need not evaluate these expressions.)

d. By simplifying the expressions in part *c.*, derive an expression for the specific heat, $C(T)$, that is valid at very low temperatures. (Hint: Include in your calculation only the two lowest-energy states.)

e. By simplifying the expressions in part *c.*, derive an expression for the specific heat, $C(T)$, that is valid at very high temperatures. (Hint: Approximate the discrete sum by an integral.)

6.

Consider a particle of charge q and mass m in constant crossed $\vec{E}$ and $\vec{B}$ fields:

$$\vec{E} = (0, 0, E) , \qquad \vec{B} = (0, B, 0) , \qquad \vec{r} = (x, y, z) .$$

a. Determine the scalar and vector potentials, $\Phi, \vec{A}$, for this field in a gauge where $\vec{A}$ has a single Cartesian component.

b. Write the Schrödinger equation for this particle, in this gauge.

c. Separate variables and reduce it to an effective 1-dimensional problem.

d. Compare this to a linear harmonic oscillator and calculate the expectation value of $\vec{r}$.

e. Calculate the expectation value of the velocity, $\vec{v}$, in any energy eigenstate, using Ehrenfest's theorem: $\frac{\mathrm{d}\langle Q \rangle}{\mathrm{d}t} = \langle \frac{\partial Q}{\partial t} \rangle - \frac{1}{i\hbar}\langle [H, Q] \rangle$.