# The Howard University
# Department of Physics and Astronomy

## Master of Science Comprehensive and Doctor of Philosophy Qualifying Exam

Part 1: *Classical Physics*
August 26, 2008

Work *four* problems, at least one from each group. Circle the numbers below to indicate your chosen problems.

| 1 | 2 3 4 | 5 6 7 8 |
|---|---|---|
| Group A | Group B | Group C |

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2. Write your code letter and a page number (in sequential order) on the *top right-hand* corner of each submitted answer sheet.
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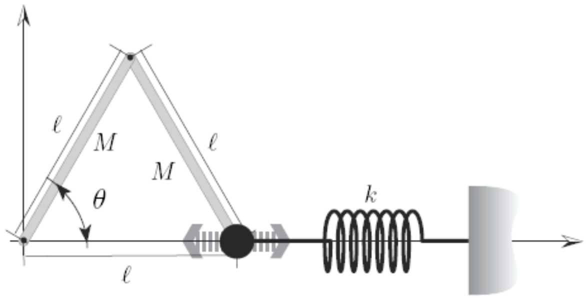
## **Group A:** *Problem #1*

Consider the arrangement of two hinged rods, each of mass $M$ and length $\ell$. The left-most end of the left-hand rod pivots around an axle at $(0, 0)$. The far end of the right-hand rod is connected to the bead of mass $m$, which is constrained to move on the horizontal $x$-axis and is initially at $(\ell, 0)$. It is also connected to the horizontal, elastic spring with spring constant $k$, and which can accommodate the motion of the bead as far as the hinge will let it, between $(0, 0)$ and $(2\ell, 0)$. As the bead moves to the right, the hinge *can* flip below the $x$-axis, and moves in the (downward) constant and homogeneous gravitational field with gravitational accelleration $g$.




*a.* Compute the kinetic energy, $KE$, of the hinge-bead system.

*b.* Compute the potential energy, $PE$, of the hinge-bead system.

*c.* From the Lagrangian, $L = KE - PE$, compute the equation of motion.

*d.* Determine the equilibrium position(s) of the bead.

*e.* Determine the frequency of small oscillations around the equilibrium position(s).

For rotation about the center of mass, $I = \frac{1}{12}M\ell^2$.

## ***Group B:*** *Problem #2*

. Write down Maxwell's equations for a medium with dielectric function $\epsilon$, permeability $\mu$, and conductivity $\sigma$.

[5pts] a. Show that there is a plane wave solution with frequency $\omega$, and obtain an expression for the complex propagation vector $\vec{k}$.

[5pts] b. Obtain an expression for the the decay constant (skin depth) of the plane wave if this medium is a good conductor at the plane wave frequency ($\sigma \geq \omega\epsilon/4\pi$).

[5pts] c. A simple model for the conductivity of this medium can be obtained by solving the equation of motion for electrons (mass $m$, charge $e$, velocity $\vec{v}$, collision frequency $\gamma$) in a field $\vec{E}_0$, in this medium:

$$m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + mg\vec{v} = e\vec{E}_0 e^{i\omega t}$$

and use $\vec{j} = en\vec{v} = \sigma\vec{E}$, where $n$ is the electron density. Using this expression, calculate the skin depth in a tenuous plasma ($\gamma$=0) in terms of $\omega_p^2$=$(4\pi ne^2)/m$.

[5pts] d. In the ionosphere (a tenuous plasma), an external static magnetic induction, $\vec{B}_0$ is present, and neglecting transverse fields, the magnetic field is in the $z$-direction. Consider the plane wave to be circularly polarized and propagating in the $z$-direction also, so that $E$=$E_0(\hat{e}_1 + i\hat{e}_2)$. The equation of motion for the electrons in the ionosphere becomes

$$m\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} \approx e\vec{E}_\pm e^{-i\omega t} + \frac{e}{c}\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\times\vec{B}_0 \ .$$

Solve this for the displacement $\vec{x}$ and hence the polarization, to show that the index of refraction is different for left- and right-circularly polarized radiation.

[5pts] e. Find the frequency at which only one circular polarization component will propagate without attenuation, in terms of $\omega_p$ and $\omega_B$=$\frac{eB_0}{mc}$

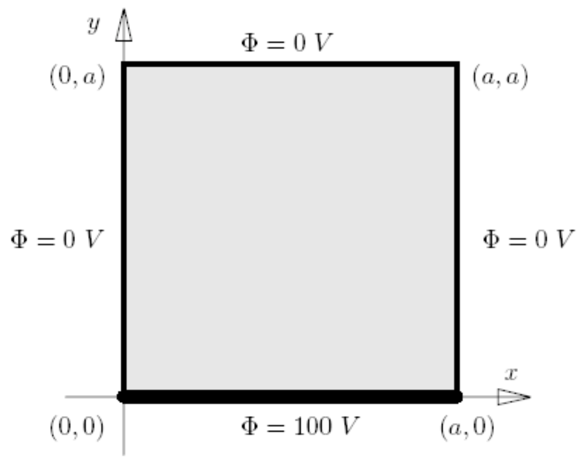
## ***Group B:*** *Problem #3*

. Consider a square, with sides of length $a$, where the electric potential $\Phi$ is 100 V on one edge of the square (i.e., along the side where $y = 0$), and $\Phi = 0$ V on the other three edges.

[15pts] a. What is the general solution for $\Phi(x, y)$ inside the square?

[10pts] b. What is the numerical value of $\Phi$ at the center of the square, that is, at the position $(\frac{1}{2}a, \frac{1}{2}a)$?

Hint: In two dimensional problems the system is considered to be very long in the $z$ direction compared to its length in the $x$ or $y$ directions. Hence, we may use the approximation that $\Phi$ is independent of $z$, i.e., $\Phi$ is a function of $x$ and $y$ only.



## ***Group B:*** *Problem #4*

A thin superconducting ring (of zero resistance, mass 50 mg, radius $r_0 = 0.5\,\mathrm{cm}$ and inductance $L = 1.3\times10^{-8}\,\mathrm{H}$) is held coaxially above a vertical, cylindrical magnetic rod. The radius of the magnetic rod is much larger than that of the ring. While the ring is held at rest, there is no current flowing through it. When it is released, it starts to fall downward, keeping its axis collinear with that of the magnet. The magnetic field produced by the rod is given by $\vec{B} = B_0\big(\beta r\hat{r} + (1-\alpha z)\hat{z}\big)$, where $B_0 = 0.01\,\mathrm{T}$, $\alpha = 2\,\mathrm{m}^{-1}$ and $\beta = 32\,\mathrm{m}^{-1}$.

[5pts] *a.* Calculate the total flux through the superconducting ring.

[5pts] *b.* Explain why the total flux through the ring must remain constant.

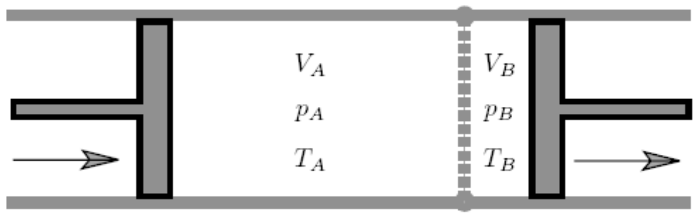
[4pts] *c.* Derive the equation of motion for the ring's vertical position, $z(t)$, and determine the numerical values of the amplitude and frequency.

[3pts] *d.* Derive an expression for the total current flowing in the ring as a function of time.

[3pts] *e.* Describe the direction of the current flow in the ring as a function of time.

## ***Group C:*** *Problem #5*

Consider a gas which undergoes an adiabatic expansion (throttling process) from region of constant pressure $p_A$ and volume $V_A$ (initially $= V_{0A}$), to a region with constant pressure $p_B$ and volume $V_B$ (initially $= 0$). The regions are separated by a porous wall, as indicated on the figure below.




[7pts] *a.* By considering the work done by the gas in the process, show that the initial and final enthalpies of the gas are equal.

[4pts] *b.* Does this conclusion apply to intermediate states? Why?

[7pts] *c.* Show that when enthalpy is conserved in a process (such as in the throttling process above), the change in the temperature, $\triangle T$, is for small pressure difference, $\triangle p$, is given by

$$\triangle T = \triangle p \left(\frac{V}{c_p}\right)(\alpha T - 1) \ ,$$

where

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p \ , \quad \text{and} \quad c_p = \left(\frac{\partial U}{\partial T}\right)_p \ .$$

[7pts] *d.* Using the above result, discuss the possibility of using the process to cool an ideal gas, and also a more realistic gas for which $p(V-b) = RT$. Explain your results.

# ***Group C:*** *Problem #6*

Consider a 2-dimensional harmonic oscillator in two dimensions, with generalized coordinates $q_\sigma$, $\sigma = 1, 2$ and two different spring constants, $k_1, k_2$.

[2pts] *a.* Write down the expression for the Hamiltonian of the system, assuming that the forces involved are conservative.

[4pts] *b.* What is the Hamilton-Jacobi equation for the Hamilton's characteristic function $W(q_1, q_2)$?

[5pts] *c.* Assuming that $W(q_1, q_2) = W_1(q_1) + W_2(q_2)$, use separation of variables to obtain separate differential equations for $W_1$ and $W_2$, respectively. (Define separation constants $\alpha_1, \alpha_2$ so that the total energy is $\alpha = \alpha_1 + \alpha_2$.)

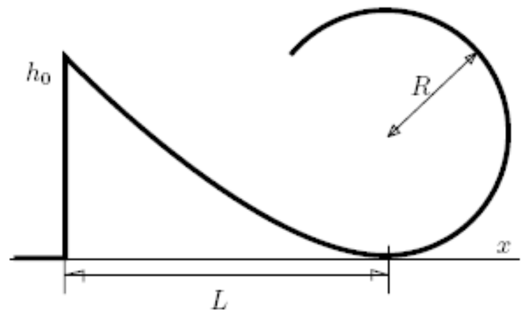
[5pts] *d.* Obtain the canonical momenta $p_\sigma$, $\sigma = 1, 2$.

[4pts] *e.* Introduce a new variable $\theta$ related to $q_\sigma$ by $q_\sigma = \sqrt{\frac{2\alpha_\sigma}{k_\sigma}} \sin\theta$ and obtain the action integral $J_\sigma$.

[5pts] *f.* Determine the fundamental frequencies $\nu_\sigma$, $\sigma = 1, 2$ of the independent modes of motion of the 2-dimensional oscillator.

## ***Group C:*** *Problem #7*

A car slides without friction down a ramp described by a height function $h(x)$, which is smooth and monotonically decreasing as $x$ increases from 0 to $L$. The ramp is followed by a smoothly connected loop of radius $R$. Gravitational acceleration is constant ($g$), and in the downward (negative $h$) direction.




[10pts] a. If the velocity is zero when $x$=0 (and $h$=$h_0$), find the minimum height $h_0$=$h(0)$ such that the car goes around the loop, never leaving the track.

[10pts] b. Consider the motion in the interval $0 < x < L$, before the loop. Assuming that the car always stays on the track, show that the velocity in the $x$-direction is related to the height as

$$\dot{x} = \sqrt{\frac{2g[h_0 - h(x)]}{1 + \left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2}} \; .$$

[5pts] c. In the particular case that $h(x) = h_0[1 - sin(\pi x/2L)]$, show that the time elapsed in going down the ramp from $(0, h_0)$ to $(L, 0)$ can be expressed as $T = (L/\sqrt{gh_0})\, f(a)$, where $a = \pi h_0/2L$, and write $f(a)$ as a definite integral. Evaluate the integral in the limiting case $h_0 \geq L$, and discuss the meaning of your answer.

# ***Group C:*** *Problem #8*

A ball drops to the floor and bounces through inelastic collisions with the floor, eventually coming to rest. The speed just after each collision is $\alpha$ ($0 < \alpha < 1$) times the speed just before the collision.

[10pts] *a.* If the ball is dropped from an initial height $h$, find the 'relaxation time' $T_r$, *i.e.*, the time it takes for the ball to come to rest on the floor.

Assume now that the ball carries a constant electric charge, $q$.

[5pts] *b.* How will the charge of the ball affect *qualitatively* its bouncing, and so the relaxation time, $T_r$?

[5pts] *c.* Derive the differential equation for the velocity of the bouncing ball before the first collision with the floor. To this end, you may consider law of the conservation of *total* energy, including the radiation of an accelerating particle. (Do not assume that the velocity or acceleration are constant!)

[5pts] *d.* Prove that the acceleration of the ball can no longer be constant.

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August 28, 2008

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## ***Group A:*** *Problem #1*

Consider a non-relativistic quantum mechanical particle of mass $m$ constrained to move freely in a two-dimensional plane within $x \in [0, L]$ and $y \in [0, 2L]$ by impenetrable (potential) walls.

*a.* Write down the Schrödinger equation for this particle, specify the boundary conditions, and write down the complete set of normalized wave-functions, $|n_x, n_y\rangle$.

*b.* Determine the energy levels, $E_{n_x,n_y}$ of this particle, as integral multiples of $E_0 = \frac{\pi^2\hbar^2}{8mL^2}$.

*c.* Tabulate the states with $E_{n_x,n_y} \leq 65E_0$, and indicate which are degenerate.

*d.* Compute the lowest (nonzero) order correction of energy of this particle due to the perturbation $\delta V = \lambda\delta(y - L)$ representing an infinitesimally thin wall, *i.e.*, $\lambda > 0$.

*e.* Compute the lowest (nonzero) order correction of energy of this particle due to the perturbation $\delta V = \lambda\delta(y - L)$ representing an infinitesimally thin crevice, *i.e.*, $\lambda < 0$.

*(Each problem is 5 points)*

## ***Group B:*** *Problem #2*

Consider an $L \times L \times L$ cube of metal, wherein the electrons may be treated as if comprised of an ideal gas confined in the cube.

[5pts] a. Write down the wave-function for the electron states and the expression for the energy levels, $E_{\vec{n}}$, where $\vec{n} = (n_x, n_y, n_z)$.

[5pts] b. Let $n = |\vec{n}|$. Determine the number of state between $n$ and $n + \mathrm{d}n$. Using the relation between $E_{\vec{n}}$ and $\vec{n}$, and $\mathrm{d}E_{\vec{n}}$ and $\mathrm{d}n$, eliminate $n$ and $\mathrm{d}n$ and obtain the number of states, $\mathrm{d}N$, within $[E, E + \mathrm{d}E]$.

[5pts] c. Determine the Fermi energy, that is, the energy of the highest occupied state.

[5pts] d. Determine the average kinetic energy of these electrons.

[5pts] e. Determine the pressure of this ideal gas of electrons.

## ***Group B:*** *Problem #3*

A given system of two *identical* particles may occupy any of the three energy levels:

$$\varepsilon_n = n\,\varepsilon, \qquad n = 0, 1, 2.$$

The lowest energy state is doubly degenerate. The system is in thermal equilibrium at temperature T. For each of the following cases, determine the *canonical partition function* and the *energy* and carefully sketch the *allowed configurations*:

a) The particles obey Fermi-Dirac statistics; [7 pts]
b) The particles obey Bose-Einstein statistics; [6 pts]
c) The (now *distinguishable*) particles obey Maxwell-Boltzmann statistics; [6 pts]
d) Discuss the condition(s) under which fermions or bosons may be treated as Boltzmann particles. [6 pts]

***Group B:*** *Problem #4*

Consider a simple particle moving in a one-dimension and under the influence of a restoring force $F = -kx^5$, $k > 0$.

a) Write down the Schrodinger equation for the wave-function y($x$) of such a particle. [7 pts]
b) Show that for very large $x$, y($x$) decays exponentially with some power of $x$; and this asymptotic wave-function $\psi_{asymp}(x) = \phi(x)$. [7 pts]
c) Writing $\psi(x) = \phi(x) f(x)$, and the differential equation that determines $f(x)$. [7 pts]
d) Consider now varying $k$ and independently $n$ for the restoring force $F = -kx^n$. Which of these two variations does (does not) preserve the qualitative features of the energy spectrum, and why (why not). [4 pts]

***Group C:*** *Problem #5*

In a particular model of the thermal behavior of crystalline solids, each of the N atoms of the solid behaves like three *independent* harmonic oscillators. The 3N harmonic oscillators, which are on *distinguishable* sites, all have the same angular frequency of vibration $\omega_0$.

(a) Write explicit expressions for the total energy E and canonical partition function Z of the solid. [7 pts]
(b) Calculate the Helmholtz free energy, F, of the solid. [7 pts]
(c) Obtain the heat capacity C(T), and to leading order, calculate the low-T and high-T limits. [7 pts]
(d) Sketch the graph of C(T) vs T. [4 pts]

## ***Group C:*** *Problem #6*

Consider an ideal gas of $N$ electrons in a volume $V$ at absolute zero. There are $V/8\pi^3$ translational states per unit volume of $k$-space.

a. Show that the Fermi energy at $T = 0$ is given by

$$\epsilon_F = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3} .$$

b. Show that the number of translational states $\epsilon(\epsilon)\mathrm{d}\epsilon$ for which the energy lies between $\epsilon$ and $\epsilon + \mathrm{d}\epsilon$ is given by

$$\rho(\epsilon)\mathrm{d}\epsilon = \frac{V}{4\pi^2}\frac{(2m)^{3/2}}{\hbar^3}\epsilon^{1/2}\mathrm{d}\epsilon .$$

c. Find the total mean energy $\bar{E}$ of this gas in terms of the Fermi energy $\epsilon_F$.

d. Show that for a fixed volume $V$, $\bar{E}$ is not proportional to the number of particles, $N$, in the container. How do you account for this result despite the fact that there is no interactions potential between the particles?.

e. Find the approximate pressure exerted by the conduction electrons in copper, on the solid lattice which confines them within the volume of the metal. Express your answer in atmospheres.

(Recall: In metals, all energy levels up to a the Fermi energy are filled. Number of electrons per unit volume = $8.4\times10^{28}$ electron/m$^3$; $\hbar = 1.05\times10^{-34}$Js; $m_e = 9.11\times10^{-31}$kg.)

*(Each part is 5 pts)*

## ***Group C:*** *Problem #7*

For a simple harmonic oscillator of mass $m$ and spring constant $k = m\omega^2$ (where $\omega$ is the classical frequency), the wave equation is

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \tfrac{1}{2}kx^2\psi = E\psi \ .$$

a. Simplify the equation to $-\psi'' + y^2\psi = 2\epsilon\psi$, where primes denote differentiation with respect to $y$: define $x = ay$ and $E = b\epsilon$ and the determine $a, b$.

b. For this simplified equation, show that $\psi$ approaches $e^{-\frac{1}{2}y^2}$ for large $y$.

c. Let $\psi = e^{-\frac{1}{2}y^2} f(y)$, and determine the differential equation for $f(y)$.

d. Show that $f$ must be either an even or an odd function of $y$.

e. Show that $f$ cannot be an infinite power series of $y$.

f. Try $f_0 = 1$, $f_1 = y$ and $f_2 = y^2 - c$ and determine $c$. Show that, corresponding to these three solutions, $\epsilon_0 = \frac{1}{2}$, $\epsilon_1 = \frac{3}{2}$ and $\epsilon_2 = \frac{5}{2}$,