The Howard University
Department of Physics and Astronomy

Master of Science Comprehensive and Doctor pf Philosophy Qualifying Exam

Written exam: Classical Physics

August 18, 2009

Attempt to solve *four* problems, at least one from each group. Circle the numbers below to indicate the problems you chose to solve:

1 2	3 4 5	6 7
Group A	Group B	Group C

1. Write in your chosen code-letter here: ☐
2. Write your code-letter and a page number (in sequential order) on the *top right-hand* corner of each submitted answer sheet.
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Problem 1.

An observer chooses a Cartesian coordinate system at the geographical altitude angle θ, such that the (x, y)-plane is tangential to the surface of the Earth, with the x-axis pointing to the North and the y-axis to the West; the z-axis points radially, away from the center of the Earth. A pendulum of length $L \ll R_{\text{Earth}}$ hangs above the origin of this coordinate system. Neglect air resistance, friction, the mass of the string and the size of the bob. Assume that gravitational acceleration g is uniform and points toward the center of the Earth, and that Earth is a perfect sphere.

[8 pt] ***a.*** Determine the equations of motion, for a non-rotating Earth, in the specified coordinate system.

[9 pt] ***b.*** For the Earth rotating with the frequency ω_0 about the North-South axis, specify the Coriolis force and the equations of motion in the specified coordinate system.

[8 pt] ***c.*** If the bob (of mass m) of the pendulum is shifted to the North a small distance $x_0 \ll L$ from its equilibrium point and released at $t = 0$, find the magnitude and direction of the pendulum's displacement (due to the rotation of the Earth) along the y-axis (East-West) in the specified coordinate system after one swing.

Problem 2.

A particle of mass m moves in a three-dimensional space under the influence of the potential $V(r) = Br^k$, where B and k are two real constants, either both positive or both negative.

[7 pt] ***a.*** Write down the Lagrangian and determine the radius r_0, of a circular orbit.

[6 pt] ***b.*** If the particle is given a tiny radial kick so that the radius oscillates around r_0, find the frequency, ω_r, of these radial oscillations.

Hint: It is simpler to introduce the effective potential and solve the radial equation of motion.

[6 pt] ***c.*** Determine the ratio (ω_r/ω_0), where $\omega_0 = \dot{\phi}$ (in standard spherical coordinates) is the frequency of the (nearly) circular motion.

[6 pt] ***d.*** Bertrand's theorem states that only two values of k lead to exactly closed orbits for all initial conditions. Determine these values, and the type of potential they represent.

Problem 3.

A horizontal, indefinitely long, negligibly thin-walled circular cylinder of radius R is split into two half-cylinders. The upper half is kept at the potential $+V_0$, whereas the lower half is at $-V_0$, for $V_0 > 0$ and with a negligibly thin insulation band in the seams.

[6 pt] ***a.*** Write down the differential equation determining the scalar electromagnetic potential, and determine the complete general solution in cylindrical coordinates adapted to this situation.

[7 pt] ***b.*** State the boundary conditions for the scalar electromagnetic potential in all space and determine as many as possible of the constants in your general solution to part ***a*** to fit these conditions.

[6 pt] ***c.*** Compute the surface charge density on the two half cylinders and explain any discontinuity.

[6 pt] ***d.*** If the two half-cylinders are separated by a small but finite distance, $0 < \delta \ll R$, compute the capacitance of this device per unit length

Hint: The result $\sum_{n=0}^{\infty} \sin\big((2n+1)\phi\big) = \frac{1}{2\sin(\phi)}$ *may be useful.*

Problem 4.

An electromagnetic plane-wave of frequency $\omega_0/2\pi$ is normally incident (along the z-axis) on the flat, plane surface of a semi-infinite isotropic and non-dispersive material of conductivity σ; assume that the frequency is low enough so as to neglect the displacement current inside the conductor, and that magnetic permeability of the the conductor is approximately that of vacuum, $\mu \approx \mu_0$.

[7 pt] ***a.*** From Maxwell's equations in media, derive the wave equation for $\vec{H} = \frac{1}{\mu}\vec{B}$.

[6 pt] ***b.*** Neglecting the displacement current, solve for $\vec{H}_{\text{in}}$ (inside the material) in the form $\vec{\varepsilon}\, e^{-z/\delta}\, e^{i(kz-\omega t)}$ and determine the "inside" polarization $\vec{\varepsilon}$ and the constants δ, k, ω in terms of the "outside" amplitude $\vec{\varepsilon}_0$, the constants ω_0, σ, and universal constants.

[6 pt] ***c.*** Compute the ratio $|\vec{H}_{\text{in}}|/|\vec{D}_{\text{in}}|$ inside the conductor, and if it is bigger or smaller than 1.

[6 pt] ***d.*** Compute the power per unit area transmitted into the conductor.

Problem 5.

Two point-charges, $+q$ and $-q$ with $q > 0$, are placed a small distance d apart.

[7 pt] ***a.*** Compute the total electrostatic potential of the two-charge system at an arbitrary point $\vec{r}$ exactly, and to the leading order in the ratio $\frac{d}{|\vec{r}|}$.

[6 pt] ***b.*** Defining $\vec{p} = q\,\vec{d}$ to be the electrostatic dipole moment and where $\vec{d}$ is the length vector from $-q$ to $+q$, compute the electrostatic field at $\vec{r}$ due to this dipole, for $|\vec{r}| \gg d$.

This dipole is placed an average height $h \gg d$ above an infinite, horizontal, grounded and perfectly conducting plane, at an angle θ between $\vec{p}$ and the normal; regard d infinitesimal from now on.

[6 pt] ***c.*** Calculate the force between the dipole and the infinite plane.

[6 pt] ***d.*** Calculate the work required to move the dipole indefinitely far from the plane.

Problem 6.

A simple model of diffusion may be obtained by considering a linear lattice, with lattice spacing a, in which a particle makes a random walk from one lattice site to an adjacent one, making random jumps at time intervals τ.

[5 pt] ***a.*** Compute the probability $P(x;N)$ that the particle has moved a distance x away from its starting position after a total of N jumps, *i.e.*, at time $t = N\tau$.

[8 pt] ***b.*** Assuming that $P(x;N)$ is peaked at $x \approx 0$ for $N \gg 1$ and using Stirling's approximation $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, compute the large-$N$ (long time) approximation for $P(x;N)$.

[7 pt] ***c.*** Normalize $P(x;N)$ so that $\int_{-\infty}^{+\infty} \mathrm{d}x\ P(x;N) = 1$.

[5 pt] ***d.*** Substituting $N \to t = N\tau$, show that the probability function $P(x,t)$ satisfies the diffusion equation, $D\frac{\partial^2 P}{\partial x^2} = \frac{\partial P}{\partial t}$, and compute the diffusion coefficient D.

Problem 7.

A vessel of volume V_1 contains N molecules of an ideal gas held at temperature T and pressure P_1. The energy of each molecule is $E_k = \frac{1}{2m}({p_x}^2 + {p_y}^2 + {p_z}^2) + \varepsilon_k$, with ε_k its k^{th} internal energy level.

[4 pt] ***a.*** Write down the partition function for this gas and justify your answer.

[7 pt] ***b.*** Compute the Helmholz free energy F, and explicitly determine the dependence on V_1 and T.

Now consider another vessel of volume V_2, containing the same number of molecules of the same ideal gas at the same temperature, but at pressure P_2.

[7 pt] ***c.*** Compute the total entropy, S, of the two vessels of ideal gas in terms of P_1, P_2, T, N.

[7 pt] ***d.*** If the vessels are connected to permit the molecules to mix without doing work, compute the change in the entropy, $\triangle S$, of the combined system and prove that it is non-negative. Specify the unique condition that makes $\triangle S = 0$.

Hint: Stirling's approximation $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ *may be useful.*

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Problem 1.

A quantum particle of mass m moves freely in one dimension, within impenetrable walls located at $x = 0$ and $x = a$.

[7 pt] ***a.*** Write down the Schrödinger equation, specify all the boundary conditions, and determine the wave-functions and the energies of all the stationary states, $|n\rangle$, where $|1\rangle$ is the state with the lowest energy, $|2\rangle$ the state with the next-to-lowest energy and so on.

[7 pt] ***b.*** If the system is, at time $t = 0$, prepared in the state $\Psi(x,0) = \frac{1}{\sqrt{2}}\big(\,|1\rangle + |2\rangle\,\big)$, compute the state $\Psi(x,t) = c_1(t)\,|1\rangle + c_2(t)\,|2\rangle$ at a subsequent time $t > 0$.

[4 pt] ***c.*** Compute the shortest time $t_* > 0$ at which both coefficients in $\Psi(x,t_*) = c_1(t_*)\,|1\rangle + c_2(t_*)\,|2\rangle$ are real, and compute their values, $c_1(t_*)$ and $c_2(t_*)$, at that time.

[7 pt] ***d.*** The wall at $x = a$ is instantly moved to $x = 2a$. Compute the probability that the system will be in the (new) ground state immediately after the move, if it was in the ground state just before the move.

Problem 2.

A quantum particle of mass m is restricted to move along the x-axis, under the influence of the potential $V(x,t) = \frac{1}{2}m\omega_0^2\,x^2 + W(t)$, where $W(t) = \frac{1}{2}\mu\omega_0^2\,\vartheta(t)\big(x - a_0\big)^2\cos(\omega t)$, with $a_0 = \sqrt{\frac{2\hbar}{m\,\omega_0}} > 0$, $\mu \ll m$, $\omega \neq \omega_0$ and $\vartheta(t) = \{\,0 \text{ for } t < 0,\ 1 \text{ for } t \geqslant 0\,\}$ is the Heaviside step-function.

[4 pt] ***a.*** Write down the Hamiltonian and the energies of the stationary states $|n\rangle$ for $t < 0$.

[8 pt] ***b.*** Compute the matrix elements $\langle n'|W(t)|n\rangle$ exactly, for all $n, n' = 0, 1, 2, \cdots$.

[8 pt] ***c.*** If the system was in the state $|0\rangle$ for $t < 0$, compute the *amplitude of probability* that the oscillator is found in the state $|1\rangle$ at $t = T > 0$, to lowest nonzero order in perturbation theory.

[5 pt] ***d.*** Compute the *probability* that the oscillator is found in the state $|1\rangle$ at $t = T > 0$, to lowest order in perturbation theory.

Problem 3.

The Hamiltonian for a 2-dimensional harmonic oscillator is $H = \frac{1}{2m}({p_x}^2 + {p_y}^2) + \frac{1}{2}m\omega^2(x^2 + y^2)$.

[6 pt] ***a.*** Determine the energy spectrum of this oscillator and its stationary states.

Consider now the effects of the coupling perturbation $H' = \lambda\, x\, y$.

[6 pt] ***b.*** Compute the correction to the energy levels, to first order in perturbation theory.

[6 pt] ***c.*** Compute the correction to the energy levels, to second order in perturbation theory.

[7 pt] ***d.*** Solve the perturbed 2-dimensional harmonic oscillator *exactly*, and compare the exact energy levels with your perturbative results above.

Hint: Changing variables from $(x, y; p_x, p_y)$ *to the standard* $(\hat{a}_x, \hat{a}_x^\dagger; \hat{a}_y, \hat{a}_y^\dagger)$ *should simplify computations.*

Problem 4.

Consider a quantum particle moving freely within a 3-dimensional potential

$$V(x, y, z) = \begin{cases} 0 & \text{if } |x| < L,\ |y| < L \text{ and } |z| < L, \\ V_0 & \text{otherwise;} \quad V_0 > 0. \end{cases}$$

[3 pt] ***a.*** Write down the Schrödinger equation and specify all boundary conditions.

[7 pt] ***b.*** Write down the general solutions to the Schrödinger equation, discuss what determines the integration constants, find the (transcendental) equation(s) that determine the energies of the stationary states, and determine if they are all bound and/or scattering states.

[5 pt] ***c.*** Let now $V_0 \to \infty$ and solve for the energies and the wave-functions of the properly normalized stationary states, $|n_x, n_y, n_z\rangle$, exactly.

[5 pt] ***d.*** Find all degenerate states with energy $6\,E_0$, $14\,E_0$ and $27\,E_0$. Considering at least these three sets, determine the symmetries of this system and if they cause all the degeneracy. Explain.

[5 pt] ***e.*** List at least four unrelated discrete symmetries of the potential. Is there an order-3 symmetry (a symmetry transformation the cube of which is the identity)? Explain.

Problem 5.

Consider the Hydrogen atom, restricted to its $n = 2$ states, under the simultaneous influence of a weak electric field $\vec{\mathcal{E}} = \mathcal{E}_0\,\hat{e}_x$, and a weak magnetic field with the vector potential $\vec{A} = \frac{1}{2}\mathcal{B}_0(x\,\hat{e}_y - y\hat{e}_x)$, and where we ignore the spin magnetic moments and any relativistic corrections.

[5 pt] ***a.*** Specify the $n = 2$ stationary states of the Hydrogen atom and list their energies.

[15 pt] ***b.*** Compute the change in the energies of the $n = 2$ states due to these perturbations, to lowest (nonzero) order in $\mathcal{E}_0, \mathcal{B}_0$.

[5 pt] ***c.*** Prove that the the matrix or the quadratic magnetic perturbation, $H'' = \frac{e^2{\mathcal{B}_0}^2}{8m_e c^2}(x^2 + y^2)$ is diagonal amongst the $n = 2$ stationary states.

You may find useful: $Y_0^0 = \frac{1}{\sqrt{4\pi}}$, $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$, $Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$, $Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$, *and* $R_{10}(r) = 2a_0^{3/2}\, e^{-r/a}$, $R_{20}(r) = \frac{1}{\sqrt{2}}a_0^{3/2}\,(1 - \frac{r}{2a})\, e^{-r/2a}$, $R_{21}(r) = \frac{1}{\sqrt{24}}a_0^{3/2}\,\frac{r}{a}\, e^{-r/2a}$ *are the first few properly normalized radial factors in* $\psi_{n\ell m}(r,\theta,\phi) = R_{n\ell}(r)Y_\ell^m(\theta,\phi)$. *Also,* $\int_0^\infty \mathrm{d}t\, e^{-t}\, t^z \equiv z!$ *for all* $z \in \mathbb{C}$.

Problem 6.

A particle of mass m and electric charge q moves freely along a circle in the (x, y)-plane, centered at the origin and with radius R.

[7 pt] ***a.*** Write down the Schrödinger equation for this quantum rotator, and determine the (properly normalized) wave-functions and energy levels of its stationary states.

The magnetic field $\vec{\mathcal{B}} = \mathcal{B}_0\,\hat{e}_z$, and with vector potential $\vec{A} = \frac{1}{2}\mathcal{B}_0(x\,\hat{e}_y - y\,\hat{e}_x)$, is now turned on.

[9 pt] ***b.*** Determine the exact wave-functions and energy levels of the stationary states. State any change in degeneracy that turning $\vec{\mathcal{B}}$ on had caused.

[9 pt] ***c.*** Compute the correction to the energies due to an additional electric field $\vec{\mathcal{E}} = \mathcal{E}_0\,\hat{e}_x$ (with the magnetic field still on), to lowest nonzero order in perturbation theory.

*Hint: Using cylindrical coordinates may be helpful; For part **b.**, determine the Hamiltonian using that the linear momentum $\vec{P}$ becomes $\vec{P} - \frac{q}{c}\vec{A}$ in the presence of the magnetic field. For part **c.**, compute the electrostatic potential.*

Problem 7.

Consider a volume V of some non-interacting, one-component quantum gas at temperature T, with molecular mass m and energy ε, and a chemical potential μ. Treat comparatively the separate cases of the gas being bosonic and fermionic.

[5 pt] ***a.*** Compute the non-relativistic energy density of quantum states, $g(\varepsilon) = \frac{\mathrm{d}n}{\mathrm{d}\varepsilon}$, by considering the number of states with total energy between ε and $\varepsilon + \mathrm{d}\varepsilon$, with $\gamma_s = 2s+1$ the spin-multiplicity.

[7 pt] ***b.*** Compute the total energy of the gas, E, as an ε-integral, separately for particles obeying the Bose-Einstein or the Fermi-Dirac statistics.

[7 pt] ***c.*** Considering this gas as a grand canonical ensemble and approximating the sum over states by an integral, relate the grand canonical potential, $-k_B T \ln(Z)$, to the total energy E, and obtain that $PV = \frac{2}{3}E$, for both the bosonic and the fermionic gas.

[9 pt] ***d.*** By expanding the integral from part a, compute the correction, $PV - Nk_B T$, to the state equation, for both bosons and fermions, as a function of N, V and T.