HOWARD UNIVERSITY
WASHINGTON, DC, 20059

DEPARTMENT OF PHYSICS AND ASTRONOMY 2355 Sixth Str., NW
(202)-806-6245 (Main Office) (202)-806-5830 (FAX)

— Quantum Mechanics —

Work four problems only. If you work on more than four problems, you *must* identify which four are to be graded.

1. Consider two operators, $\hat{A}$ and $\hat{B}$ which satisfy:

$$\big[\,\hat{A}\,,\,\hat{B}\,\big] = \hat{B}\ , \qquad \hat{B}^{\dagger}\hat{B} = \mathbb{1} - \hat{A}^2\ , \qquad \hat{A}\,|a\rangle = a\,|a\rangle\ \ .$$

a. Determine the hermiticity properties of $\hat{A}$ and $\hat{B}$.

b. Using the fact that $|a = 0\rangle$ is an eigenstate of $\hat{A}$, construct the other eigenstates of $\hat{A}$.

c. Suppose the eigenstates of $\hat{A}$ form a complete set. Determine if eigenstates of $\hat{B}$ can be constructed, and if so, determine the spectrum of the eigenstates of $\hat{B}$.

2. Consider a 2-dimensional harmonic oscillator, for which the Hamiltonian can be written as $H = \frac{1}{2m}({p_x}^2 + {p_y}^2) + \frac{1}{2}m\omega^2(x^2 + y^2)$.

a. Write down energies of the allowed states of this oscillator (in units of $\hbar\omega$) and specify their degeneracy.

b. For a suitable constant α, does a perturbation of the form $V = \alpha x$ change the degeneracy? Why (why not)?

c. For a suitable constant β, does a perturbation of the form $V = \beta x^2$ change the degeneracy? Why (why not)?

d. For a suitable constant γ, does a perturbation of the form $V = \gamma x^4$ change the degeneracy? Why (why not)?

e. Use perturbation theory to calculate the first order shift in the ground state energy, caused by a small perturbation $V = \gamma x^4$.

f. For all of the above perturbations and for any arbitrary collection of states, is it necessary to use *degenerate* perturbation theory? Why (why not)?

3. A particle of mass m moves in 3-dimensional space under the influence of the ("opaque bubble") potential of the form $V(r) = -\gamma\,\delta(r-a)$, for a, γ positive constants.

a. Describe the general form of the spectrum. For which values of the energy is the spectrum discrete, and for which values is it continuous?

b. Write down the Schrödinger equation in spherical coordinates, and obtain the radial equation for $u_{E,\ell}(r)$, assuming that the wave function can be written as $\Psi(r,\theta,\phi) = r^{-1}\,u_{E,\ell}(r)\,Y_\ell^m(\theta,\varphi)$, where $Y_\ell^m(\theta,\varphi)$ are the spherical harmonics.

c. Describe the S-wave solutions (ℓ=?). Sketch their radial function $u_{E,\ell}(r)$, and specify all the boundary/matching conditions.

d. From the boundary/matching conditions, find the transcendental equation which determines the energy quantization (for the discrete part of the spectrum), for $\ell = 0$.

e. Determine the "translucent" limit, *i.e.*, the smallest value of γ for which there is precisely one bound state. Find a lowest order estimate for the energy of this state.

(In spherical coordinates, $\nabla^2\Psi = \frac{\partial^2\Psi}{\partial r^2} + \frac{2}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\big(\sin\theta\,\frac{\partial\Psi}{\partial\theta}\big) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\varphi^2}$.)

4. Examine the pionic decays of K^0 governed by weak interactions. The dominant decays are $K^0 \to \pi^+\pi^-$ and $K^0 \to \pi^0\pi^0$. Total isospin is *not* conserved in these processes, but changes either by $\triangle I = +\frac{1}{2}$ or by $\triangle I = -\frac{1}{2}$.

a. Introducing appropriate creation and annihilation operators, write down the interaction terms ($\hat{H}_I$) in the Hamiltonian corresponding to the given decay processes. Given the basic particle data for K^0 and $\pi^0, \pi^\pm$ (given below), does $\hat{H}_I$ commute with isospin vector-operator $\hat{\vec{I}}$? Why (why not)?

b. Using the basic particle data (below), determine the possible values of the angular momentum and isospin of each of the two-pion system into which the K^0 decays.

c. Based only on the particle data below, estimate the ratio of cross-sections $\frac{\sigma(K^0 \to \pi^+\pi^-)}{\sigma(K^0 \to \pi^0\pi^0)}$.

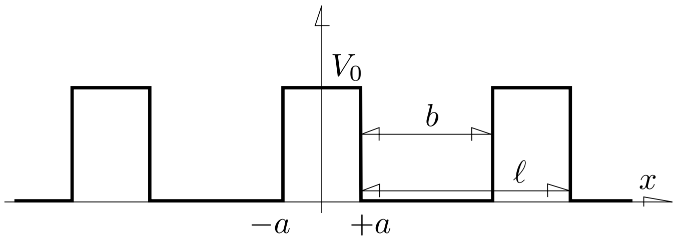
d. If the charge-conjugation & parity (reflection) operator acts as $\mathrm{CP}\big|K^0\big\rangle = |\overline{K^0}\rangle$ and obeys $(\mathrm{CP})^2 = \mathbb{1}$, find the ortho-normalized kaon (K) CP-eigenstates.

e. For $|K_\pm\rangle$, such that $\mathrm{CP}|K_\pm\rangle = \pm|K_\pm\rangle$, let $\Gamma_\pm$ denote the decay rate. What is the fraction of K^0's in an initially pure K^0-beam, as a function of proper time?

f. Is the decay $K^0 \to \pi^+\pi^0\pi^-$ possible? Is the decay $K^0 \to \pi^+\pi^0\pi^0\pi^-$ possible? Prove your assertions by a short calculation.

The mesons, $K^0, \pi^\pm, \pi^0$, have no spin and are odd under parity (space reflection). The $K^0, \overline{K^0}$ have (isospin) $I_z = -\frac{1}{2}, +\frac{1}{2}$, respectively, while the π^+, π^0, π^- have $I_z = +1, 0, -1$, respectively. The (rest-) masses are (in MeV/c^2): $m_{K^0} = 497.7$, $m_{\pi^0} = 135.0$, $m_{\pi^\pm} = 139.6$. Some possibly useful Clebsh-Gordan coefficients: $\langle 1,1;1,-1|2,0\rangle = 1/\sqrt{6}$, $\langle 1,1;1,-1|1,0\rangle = 1/\sqrt{2}$, $\langle 1,1;1,-1|0,0\rangle = 1/\sqrt{3}$, $\langle 1,1;0,0|2,0\rangle = \sqrt{2/3}$, $\langle 1,1;0,0|1,0\rangle = 0$, $\langle 1,1;0,0|0,0\rangle = -1/\sqrt{3}$.

5. Consider a particle of mass μ in a 1-dimensional periodic potential shown in the figure. The height of the barriers is V_0, and the potential satisfies $V(x+\ell) = V(x)$:



a. Using the translational symmetry, prove that there is a complete set of stationary states which obey

$$\psi_{_E}(x+\ell) = e^{iK\ell}\psi_{_E}(x) \ , \qquad E > 0 \ ,$$

where K is a constant.

b. Determine this K by imposing a periodic boundary condition on the wave function over a large but finite region with N barriers: $\psi_{_E}(x+N\ell) \equiv \psi_{_E}(x)$, for all E.

c. Write down the general solution within $-a \leq x \leq \ell{+}a$, and for any $E > 0$. Using the results from parts a. and b., reduce the number of undetermined constants to four. Then use the boundary matching conditions to find the system of equations which determins the remaining four constants (you need not solve this system).

d. When $0 < E < V_0$, and writing $k\hbar = \sqrt{2mE}$ and $\kappa\hbar = \sqrt{2m(V_0{-}E)}$, the energy condition

$$\cos(kb)\cosh(2\kappa a) + \frac{\kappa^2 - k^2}{2\kappa k}\sin(kb)\sinh(2\kappa a) = \cos(K\ell)$$

must be enforced for $\psi_{_E} \not\equiv 0$. Show that this forbids certain regions of energy.

e. Considering carefully the limit when $a \to 0$ and $V_0 \to \infty$ but $2aV_0 = \Omega = \textit{constant}$, find the resulting energy condition and obtain the lowest order estimate for the minimum allowed energy.

6. The spin Hamiltonian for a spin-$\frac{1}{2}$ particle in a magnetic field is given by $H = \vec{\omega}\cdot\vec{S}$, where the matrix representation of the spin operators is given by $[\vec{S}] = \frac{1}{2}\hbar\vec{\boldsymbol{\sigma}}$, in terms of the usual Pauli matrices $\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2, \boldsymbol{\sigma}^3$.

a. Show that the time evolution operator for the quantum state vectors takes the form $\boldsymbol{U}(t) = e^{i\mathbf{M}t}$, where $\mathbf{M}^2 = \mathbb{1}$.

b. Expand the time evolution matrix to show that it is proportional to a linear combination of $\mathbb{1}$ and $\mathbf{M}$.

c. If the system is in the state $\left|m = +\frac{1}{2}\right\rangle$ at time $t = 0$, determine the state of the system at a later time $t > 0$.

d. Determine the probability that the system is measured to be in state $\left|m = +\frac{1}{2}\right\rangle$ at a later time $t > 0$.