The Howard University
Department of Physics and Astronomy

Master of Science Comprehensive and
Doctor of Philosophy Qualifying Exam

Written exam: Classical Physics
August 23, 2011

Attempt to solve four problems, at least one from each group. Circle the numbers below to indicate the problems you chose to solve:

1 2 3 4 5 6 7
Group A Group B Group C

1. Write in your chosen code-letter here: [ ]
2. Write your code-letter and a page number (in sequential order) on the top right-hand corner of each submitted answer sheet.
3. Write only on one side of your answer sheets.
5. Stack your answer sheets by problem and page number, and then staple them (at the top left-hand corner) with this cover sheet on the top.
Problem 1.
A bead of mass $m$ is free to slide along a frictionless massless hoop of radius $R$, in a uniform vertical gravitational field. The hoop rotates with constant angular speed $\omega$ around a vertical diameter, as in the illustration to the right.

[9 pt] **a.** Write the Lagrangian of the mass.

[8 pt] **b.** Find the equations of motion for the angle $\theta$, defined as shown in the illustration.

[8 pt] **c.** What are the equilibrium positions? Are stable or unstable? Discuss the cases $\omega^2 < g/R$ and $\omega^2 > g/R$.

Problem 2.
Consider a particle of mass $m$ constrained to move on the surface of a cylinder of radius $R$. The particle is subjected to a force directed toward the origin and proportional to the distance from the origin, $F = -kr$.

[8 pt] **a.** Find the Hamiltonian of the particle.

[6 pt] **b.** Discuss the occurrence of cyclic coordinates and constants of motion.

[6 pt] **c.** Obtain the canonical equations.

[5 pt] **d.** Find the equations of motion.

Problem 3.
$N$ weakly-coupled particles obeying Maxwell-Boltzmann statistics may each exist in one of three non-degenerate energy levels of energies $-\epsilon, 0$ and $\epsilon$. The system is in contact with a thermal reservoir at temperature $T$.

[5 pt] **a.** What is the entropy of the system at $T = 0$ K?

[4 pt] **b.** What is the maximum possible entropy of the system?

[4 pt] **c.** What is the minimum possible energy of the system?

[4 pt] **d.** What is the canonical partition function of the system?

[4 pt] **e.** What is the most probable energy of the system?

[4 pt] **f.** If $C(T)$ is the heat capacity of the system, what is the value of $\int_0^\infty \frac{C(T)}{T} dT$?
Problem 4.

Consider a collection of \( N \) two-level systems in thermal equilibrium at a temperature \( T \). Each system has only two states: a ground state of energy 0 and an excited state of energy \( \epsilon \).

\[ \text{[7 pt]} \ a. \text{ Obtain the probability that a given system will be found in the excited state.} \]

\[ \text{[6 pt]} \ b. \text{ Sketch the temperature dependence of this probability.} \]

\[ \text{[7 pt]} \ c. \text{ Find the entropy of the entire collection of two-level systems.} \]

\[ \text{[5 pt]} \ d. \text{ Sketch its temperature dependence.} \]

Problem 5.

Consider an adsorbent surface having \( N \) sites, each of which can adsorb one gas molecule. Suppose the surface is in contact with an ideal gas with the chemical potential \( \mu \) (determined by pressure \( p \) and temperature \( T \)). Assume that an adsorbed molecule has energy \(-\epsilon_0\) compared to one in a free state.

\[ \text{[5 pt]} \ a. \text{ Find the number of possible ways of distributing } N_1 \text{ identical molecules among } N \text{ adsorbing sites.} \]

\[ \text{[4 pt]} \ b. \text{ Write down the canonical partition function } Z_{N_1}. \]

\[ \text{[4 pt]} \ c. \text{ Obtain the grand partition function } Q. \]

\[ \text{[4 pt]} \ d. \text{ What is the probability } Pr(N_1) \text{ of } N_1 \text{ molecules being adsorbed?} \]

\[ \text{[4 pt]} \ e. \text{ Obtain the mean value } N_1 \text{ of the number of adsorbed molecules.} \]

\[ \text{[4 pt]} \ f. \text{ Determine the covering ratio } \theta = (N_1 / N). \]

Problem 6.

Consider a long cylindrical wire of radius \( a \) carrying a steady uniform current \( I \) in the \( \hat{e}_z \) direction. The conductivity of the material is \( \sigma \). (Briefly indicate your choice of coordinates.)

\[ \text{[5 pt]} \ a. \text{ Find the electric field vector } \vec{E} \text{ inside the wire.} \]

\[ \text{[5 pt]} \ b. \text{ Find the magnetic field } \vec{B} \text{ inside and outside the wire.} \]

\[ \text{[5 pt]} \ c. \text{ Find the Poynting vector inside and (just barely) outside the wire; assume the electric field just outside the wire is the same as the field inside.} \]

\[ \text{[5 pt]} \ d. \text{ Find the Poynting vector at the surface of the wire, and the energy flux through the surface, for a segment of length } L. \]

\[ \text{[5 pt]} \ e. \text{ Compare the result of part (d) with the Joule heating (i.e. power dissipated) in the wire and comment on the physical significance.} \]
Problem 7.

Consider a uniform plane wave in a vacuum whose electric field takes the following form:

\[
\vec{E}(\vec{r}, t) = E_0 (3\hat{e}_x + a\hat{e}_y) \cos \left( \frac{x - y + 2z}{L} - \omega t \right)
\]  

(1)

**a.** In terms of \(L\): what is the wavelength \(\lambda\)? What is the angular frequency \(\omega\)?

**b.** Write down the vacuum Maxwell equations. Use them to determine \(a\).

**c.** Use the vacuum Maxwell equations to determine the magnetic field \(\vec{B}\) of this plane wave.

**d.** What are the electric and magnetic energy densities?

**e.** What is the Poynting vector for this wave?
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Problem 1.

Consider a dilute diatomic gas whose N molecules consist of non-identical pairs of atoms. The moment of inertia about an axis through the molecular center-of-mass, perpendicular to the line connecting the two atoms, is \( I \).

\[ a. \] Write an expression for the rotational energy \( \varepsilon_J \) of a single molecule in terms of the quantum number \( J \).

\[ b. \] Obtain the total canonical partition function \( Z \) for the system.

\[ c. \] Determine the rotational energy \( U_R \), as well as contributions to the heat capacity \( C_R \) and the entropy \( S_R \) per mole at temperature \( T \).

\[ d. \] Derive expressions for \( U_R, C_R \) and \( S_R \) for two limiting cases: (i) \( k_B T \gg \hbar^2 / I \) (ii) \( k_B T \ll \hbar^2 / I \).

Problem 2.

A particle of mass \( M \) moves in one dimension, where it is bound by an infinite potential well of width \( L \) that is divided into two equal parts, as illustrated to the right. On the left side the potential is zero, on the right side the potential is \( V_0 > 0 \). Everywhere else the potential is infinite. The initial wave function for the particle is given by the form

\[ \psi(x) = \begin{cases} A_1 \sin(8\pi x / L) & \text{for } -\frac{L}{2} < x < 0 \\ A_2 \sin(4\pi x / L) & \text{for } 0 < x < \frac{L}{2}. \end{cases} \]

\[ a. \] Under what condition(s) on \( V_0, M \) and \( L \) is this an eigenfunction of the energy for the 1D Schrödinger equation?

\[ b. \] Determine the values of \( A_1 \) and \( A_2 \) that give a normalized energy eigenfunction of the given form.

\[ c. \] What is its energy?

\[ d. \] What is the probability that the particle is in the right half of the well?
Problem 3.

A rigid, diatomic molecule with moment of inertia $I$ is known to be in a state

$$\psi(r, \theta, \phi, 0) = \frac{R(r)}{\sqrt{26}} \left( 3Y_1^1(\theta, \phi) + 4Y_2^3(\theta, \phi) + Y_3^1(\theta, \phi) \right)$$

where $R(r)$ is a fixed (and normalized) radial function and $Y_l^m(\theta, \phi)$ are the usual spherical harmonics.

[10 pt] **a.** Calculate the measured values of $L = \sqrt{L^2}$ and $L_z$ in this state and the probabilities with which these values occur.

[8 pt] **b.** Calculate $\psi(r, \theta, \phi, t)$ at $t > 0$.

[7 pt] **c.** Calculate the expectation value of energy, $\langle E \rangle$, for this molecule.

Problem 4.

Consider a particle of mass $m$ restricted to move in one dimension, bound by the potential $V(x) = -\frac{\lambda}{x}$ for $x > 0$ and $V(x) = +\infty$ for $x \leq 0$, with $\lambda > 0$ a real appropriate constant.

[4 pt] **a.** Write down the Schrödinger equation and specify/state all the boundary conditions.

[5 pt] **b.** Motivated by the Hydrogen atom, determine $\alpha$ so that $\psi_{\infty}(x) = e^{-x/\alpha}$ would approximately solve the Schrödinger equation when $x \gg \alpha$.

[10 pt] **c.** Writing $\psi(x) = f(x)e^{-x/\alpha}$, determine the differential equation for $f(x)$. Show that physically acceptable solutions must be polynomials with $f(0) = 0$, and then compute the bound state energy levels ($E < 0$) for which $\psi(x)$ satisfies all boundary conditions.

[6 pt] **d.** Compute $\langle \psi | x | \psi \rangle$ for $\psi_0(x) = N x e^{-x/\alpha}$, with the value of $\alpha$ you computed in part b), and $N$ the proper normalization constant. Is $\psi_0(x)$ a stationary state? Is there a stationary state of energy lower than $\psi_0(x)$? Justify your answers.
Problem 5.

A 3-dimensional harmonic oscillator of frequency $\omega$, is placed in a weak electric field $\vec{E} = E_0 \hat{e}_z$, and the harmonic oscillator itself has the electric charge $q$.

7 pt  a. In the $E_0 \to 0$ limit and using Cartesian coordinates, specify the energies of the stationary states of this oscillator, as well as the corresponding state-vectors using the creation/annihilation operators in the $x$-, $y$- and $z$-direction.

3 pt  b. Specify the degeneracies of at least the lowest-lying five energy levels.

6 pt  c. Compute, to lowest non-vanishing perturbative order, the correction to all energy levels of this oscillator stemming from the $\vec{E}$-field.

6 pt  d. Compute the correction to all energy levels of this oscillator stemming from the $\vec{E}$-field exactly, and compare with your perturbative result.

3 pt  e. Does the $\vec{E}$-perturbed oscillator still have any degeneracy? Explain.

Problem 6.

Consider a spin-$1/2$ quantum system in a mixed state represented by the state operator $\hat{\rho}$.

10 pt  a. Show that the most general form of $\hat{\rho}$ is $\frac{1}{2}(a_0 \mathbb{1} + \vec{a} \cdot \vec{\sigma})$, where $\vec{\sigma} = \sum_{i=1}^{3} \sigma_i \hat{e}_i$ and $\sigma_i$ are the Pauli matrices. Show that $a_0 = 1$ and $\vec{a}$ is a real vector with $\vec{a}^2 \leq 1$.

5 pt  b. If the magnetic moment of this system is $\vec{\mu} = \frac{1}{2} \gamma \hbar \vec{\sigma}$, specify the energy (Hamiltonian) that a constant magnetic field, $\vec{B}$, adds to this system.

10 pt  c. Calculate the time-dependent state-operator $\hat{\rho}(t)$ in the Shrödinger picture, i.e., describe the effect of the magnetic field on this system in terms of the time-evolution of $\vec{a} = \vec{a}(t)$.

(Hint: $\sigma_j \sigma_k = \delta_{jk} \mathbb{1} + i \epsilon_{jk}^\ell \sigma_\ell$ for $j, k, \ell = 1, 2, 3$ may be useful; $\epsilon_{jk}^\ell$ is the Levi-Civita alternating symbol.)