The Howard University
Department of Physics and Astronomy

Master of Science Comprehensive and Doctor of Philosophy Qualifying Exam

Written exam: Classical Physics

August 21, 2012

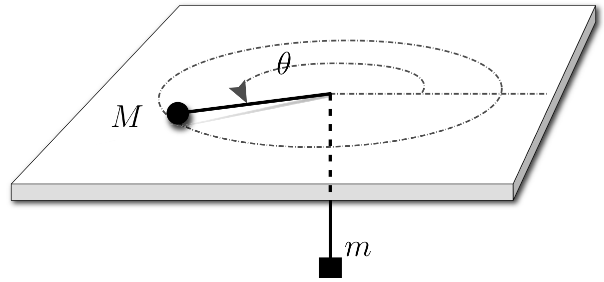
Attempt to solve ***four*** problems, at least one from each group. Circle the numbers below to indicate the problems you chose to solve:

1 2	3 4	5 6 7
Group A	Group B	Group C

1. Write in your chosen code-letter here: ☐
2. Write your code-letter and a page number (in sequential order) on the *top right-hand* corner of each submitted answer sheet.
3. Write only on one side of your answer sheets.
4. Start each problem on a new answer sheet.
5. Stack your answer sheets by problem and page number and then staple them (at the top *left-hand corner*), with this cover sheet on the top.

Problem 1.

A particle of mass M is constrained to move on a horizontal plane. A second particle of mass m, is constrained to a vertical line. The two particles are connected by a massless string of length L which passes through a hole in the plane (see figure). Neglect all friction.



[6 pt] ***a.*** Find the Lagrangian of the system.

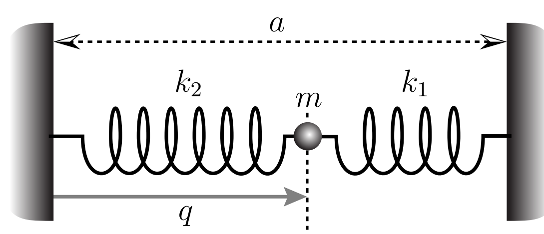
[6 pt] ***b.*** Derive the equations of motion.

[6 pt] ***c.*** Determine the constants of the motion.

[7 pt] ***d.*** For a particular initial condition, the mass M will move along a stable circle of radius r_0. Find the value of r_0 in terms of the parameters of the problem and constants of motion.

Problem 2.

A particle of mass m can move in one dimension under the influence of two springs connected to fixed points at a distance a apart (see figure). The springs obey Hooke's law and have zero unstretched lengths and force constants k_1 and k_2, respectively.



[5 pt] ***a.*** Using the position q of the particle from one fixed point as the generalized coordinate, find the Lagrangian and the corresponding Hamiltonian.

[5 pt] ***b.*** Determine and explain if the energy conserved, and if the Hamiltonian is conserved.

Introduce a new coordinate Q defined by $Q = q - b \sin \omega t$, with $b = k_2 a/(k_1 + k_2)$.

[5 pt] ***c.*** Determine the Lagrangian in terms of Q.

[5 pt] ***d.*** Determine the corresponding Hamiltonian.

[5 pt] ***e.*** Determine and explain if the energy conserved, and if the Hamiltonian is conserved.

Problem 3.

Inside a superconductor, instead of Ohm's Law ($\mathbf{J} = \sigma\mathbf{E}$), we assume London's equations to be valid for the current density $\mathbf{J}$:

$$c\boldsymbol{\nabla}\times(\lambda\,\mathbf{J}) = -\mathbf{B} \quad \text{and} \quad \frac{\partial}{\partial t}(\lambda\,\mathbf{J}) = \mathbf{E}$$

(in Gaussian units), where λ is a constant. Otherwise, Maxwell's equations (with $\epsilon = 1, \mu = 1$ outside the slab) and the corresponding boundary conditions are unchanged.

Consider an infinite superconductor slab of thickness $2d$ ($-d \leqslant z \leqslant d$), outside of which there is a given constant magnetic field parallel to the surface, $H_x = 0 = H_z$, and $H_y = H_0$ above and below the slab (*same* value for $z > d$ and $z < -d$), and with $\mathbf{E} = \mathbf{D} = 0$ everywhere.

[9 pt] ***a.*** If surface currents and charges are absent, compute $\mathbf{B}$ inside the slab.

[8 pt] ***b.*** If surface currents and charges are absent, compute $\mathbf{J}$ inside the slab.

[8 pt] ***c.*** Compute the magnetization $\mathbf{M}$ inside the slab.

Problem 4.

Two point charges, $+q$ and $-q$ with $q > 0$, are placed at a small distance d apart.

[7 pt] ***a.*** Compute the total electrostatic potential of the two-charge system at an arbitrary point $\mathbf{r}$ exactly, and to the leading order in the ratio $d/|\mathbf{r}|$.

[6 pt] ***b.*** Defining $\mathbf{p} = q\mathbf{d}$ to be the electrostatic dipole moment and where $\mathbf{d}$ is the length vector from $-q$ to $+q$, compute the electrostatic field at $\mathbf{r}$ due to this dipole, for $|\mathbf{r}\,| \gg |\mathbf{d}| = d$.

This dipole is placed at an average height $h \gg d$ above an infinite, horizontal, grounded and perfectly conducting plane, at an angle θ between $\mathbf{p}$ and the vertical; regard d as infinitesimal from now on.

[7 pt] ***c.*** Calculate the force between the dipole and the infinite plane.

[5 pt] ***d.*** Calculate the work required to move the dipole indefinitely far from the plane.

Problem 5.

A simple theory of thermodynamics of a ferromagnet uses the Gibbs free energy G as a function of the magnetization M in the following form:

$$G = G_0 - HM + A(T - T_c)M^2 + BM^4$$

where H is the magnetic field, G_0, A and B are positive constants, T is the temperature and T_c is the critical temperature.

[9 pt] ***a.*** What condition must be imposed on the free energy G to determine the thermodynamically most probable value of the magnetization M in equilibrium?

[8 pt] ***b.*** Determine the equilibrium value of M for $T > T_c$ and sketch a graph of M *vs.* T for small and constant H.

[8 pt] ***c.*** Comment on the physical significance of the temperature dependence of M as T reaches close to T_c for small H in case (b).

Problem 6.

A perfect gas is defined as one whose equation of state is given by $PV = Nk_BT$ and whose internal energy U is only a function of the temperature T. Here, P is the pressure, V is the volume, N is the number of molecules, T is the absolute temperature of the gas and k_B is the Boltzmann constant.

[5 pt] ***a.*** Define the specific heats, C_P and C_V, in terms of the entropy of the system.

[10 pt] ***b.*** Show that $C_P - C_V = Nk_B$.

[10 pt] ***c.*** Show that the quantity PV^γ is constant during an adiabatic expansion, where the constant γ is the ratio of molar specific heats.

Problem 7.

Start from the thermodynamic relation $dF = -SdT - PdV$.

[8 pt] ***a.*** Derive the following Maxwell relationship, $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$.

[9 pt] ***b.*** Based on his famous electromagnetic theory, Maxwell found that the pressure P exerted by isotropic radiation in nature is related to the energy $U(T)$ via the following equation: $P = U(T)/(3V)$, where V is the volume of the relevant radiation cavity. Use your result of part (a) and the first and second laws of thermodynamics to obtain the following relation for the energy density, $u(T) = U(T)/V$: $3u(T) = T\left(\frac{\partial u}{\partial T}\right) - u$.

[8 pt] ***c.*** Solve the equation derived in part (b) and ontain Stefan's law of blackbody radiation relating u and T explicitly.

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Problem 1.

A non-relativistic particle of mass m moves in a three-dimensional central potential $V(r)$ which vanishes at $r \to \infty$. We are given that an exact eigenstate is $\psi(r) = C\, r^{\sqrt{3}} e^{-\alpha r} \cos(\theta)$, where C and α are real, positive constants.

[7 pt] ***a.*** Determine the angular momentum of this state.

[6 pt] ***b.*** Determine the energy of this state.

[6 pt] ***c.*** Determine the normalization constant C.

[6 pt] ***d.*** Determine the potential $V(r)$.

Problem 2.

A particle of mass m is restricted to move along the x-axis and has been observed to be in a stationary state described by the wave-function $\psi_k(x) = A \tanh(kx)$, where k is a positive constant and A is an appropriate amplitude.

[4 pt] ***a.*** Specify the probability density for this state, state if this is a bound or a scattering state, and specify the condition that should be used to determine the value of A.

[6 pt] ***b.*** Knowing that $\psi_t(x)$ satisfies an appropriate Schrödinger equation, determine the potential $V(x)$ and energy E for which this will be true, such that $\lim_{x\to\pm\infty} V(x) = 0$.

[6 pt] ***c.*** Determine if the same particle, under the influence of the same potential can also be found in the stationary state $\phi_k(x) = B\frac{1}{\cosh(kx)}$, and what its energy should be if it can.

[4 pt] ***d.*** Specify the probability density for $\phi_k(x)$, state if this is a bound or a scattering state, and specify (and solve if possible) the condition that determines the value of B.

[5 pt] ***e.*** Using the WKB approximation, find the integral condition for the energies of bound states, without evaluating it. Use this to estimate how may bound states can this potential have.

(Possibly useful: $\cosh^2(x) - \sinh(x) = 1$, $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\cosh(x)} = \pi$, $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\cosh^2(x)} = 2$.)

Problem 3.

Consider a 1-dimensional harmonic oscillator with the characteristic frequency ω, specified in terms of the standard creation and annihilation operators, $\hat{a}$ and $\hat{a}^\dagger$, for which $[\hat{a}, \hat{a}^\dagger] = 1$.

[5 pt] ***a.*** Given the Hamiltonian $\widehat{H} = \hbar\omega(\hat{a}^\dagger\hat{a}+\frac{1}{2})$, determine the result of $\hat{a}\,|n\rangle$ and $\hat{a}^\dagger\,|n\rangle$ for the standard eigenstates $\widehat{H}\,|n\rangle = \hbar\omega(n+\frac{1}{2})\,|n\rangle$, using only the given information.

[5 pt] ***b.*** With $\widehat{K}_+ = (\hat{a}^\dagger)^2$, $\widehat{K}_- = (\hat{a})^2$ and $\widehat{K}_0 = \hat{a}^\dagger\hat{a}+\frac{1}{2}$, compute the commutators $[\widehat{K}_+, \widehat{K}_-]$ and $[\widehat{K}_0, \widehat{K}_\pm]$ to show that $[\widehat{K}_i, \widehat{K}_j] = \sum_k f_{ij}{}^k \widehat{K}_k$ with $i, j, k = -, 0, +$ and compute all constants $f_{ij}{}^k$. Compare with the known case of angular momentum operators.

[5 pt] ***c.*** Show that the action of $\widehat{K}_\pm$ on the states $|n\rangle$ is consistent with your results in part **b**.

[5 pt] ***d.*** Determine the eigenstate $\hat{a}\,|z\rangle = z\,|z\rangle$ as the expansion $|z\rangle := \sum_n c_n\,|n\rangle$ by determining the coefficients recursively.

[5 pt] ***e.*** Prove that both $|z\rangle = e^{z\hat{a}^\dagger - z^*\hat{a}}\,|0\rangle$ and $|z\rangle = e^{-\frac{1}{2}|z|^2}\,e^{z\hat{a}^\dagger}\,|0\rangle$.

(Possibly useful: $[\hat{X}, f(\hat{Y}, \hat{Z})] = [\hat{X}, \hat{Y}]\,\frac{\partial f}{\partial \hat{Y}} + [\hat{X}, \hat{Z}]\,\frac{\partial f}{\partial \hat{Z}}$; also, operator functions are *defined* by their Taylor expansions.)

Problem 4.

A 2-dimensional isotropic harmonic oscillator in the (x, y)-plane with characteristic frequency ω, mass m and electric charge q is placed in a constant magnetic field $\mathbf{B} = B_0\hat{\mathbf{e}}_z$.

[6 pt] ***a.*** Using the vector potential $\mathbf{A} = \frac{1}{2}(\mathbf{B}\times\mathbf{r})$ and that coupling with the magnetic field modifies the linear momentum into $\mathbf{p} \to \mathbf{p} + \frac{q}{c}\mathbf{A}$, determine the full Hamiltonian for this oscillator in polar coordinates, and the appropriate Schrödinger equation. Specify the physical consequences of the coupling to the external magnetic field in terms of typical physical characteristics (energy, angular momentum, frequency, *etc.*)

[10 pt] ***b.*** For $B_0 = 0$ and using polar coordinates, determine the (un-normalized) stationary solutions in the form $\Psi(\rho, \phi) = e^{in\phi}\,e^{-\alpha\rho^2} f_n(\rho)$, the possible values for n and the energies of this oscillator, and compare with the solutions in Cartesian coordinates.

[4 pt] ***c.*** For $B_0 = 0$, determine the degeneracies of at least the lowest-lying five energy levels.

[5 pt] ***d.*** For $B_0 \neq 0$, compute the exact energy levels of this oscillator.

(For part **d**, your may benefit from judiciously adapting the computations from part **b**.)

Problem 5.

Consider a quantum particle of mass m constrained to move freely in one dimension, along $x \in [0, L]$, but incapable of moving outside of this box.

[8 pt] ***a.*** Write down the Schrödinger equation governing the dynamics of this particle, specify all boundary conditions, and use them to completely determine the complete set and the energies of the stationary states.

[3 pt] ***b.*** Compute the correct normalization for all stationary states.

[5 pt] ***c.*** Consider the effects of a perturbation of the form $\widehat{H}' = \lambda\delta(x - \frac{1}{3}L)$. Integrating the Schrödinger equation between $\frac{1}{2}L \pm \epsilon$ when $\epsilon \to 0$, derive the matching conditions at $x = \frac{1}{3}L$.

[4 pt] ***d.*** Compute the correction to the energies of the stationary states to lowest non-vanishing order in perturbation theory.

[5 pt] ***e.*** Specify the effect on the particle of the perturbation becoming an impenetrable wall, either by considering the $\lambda \to \infty$ limit of your previous results or by re-considering the boundary conditions.

Problem 6.

Consider a quantum system with the Hamiltonian $\widehat{H}_0$, which has only two stationary states, $|1\rangle \neq |2\rangle$, with $E_1 \neq E_2$.

[6 pt] ***a.*** Determine the most general time-dependent state for this system as the most general solution of the appropriate Schrödinger equation.

[6 pt] ***b.*** Compute the probability that at a time $t > 0$ this system is found to be in the particular state $|\Psi_{\overline{\alpha}}\rangle = \cos(\alpha)\,|2\rangle - \sin(\alpha)\,|1\rangle$, if it it started off as $\Psi_\alpha(0) = \cos(\alpha)\,|1\rangle + \sin(\alpha)\,|2\rangle$. Determine the angle α which maximizes the oscillations in the probability $P_{\alpha\to\overline{\alpha}}$.

[6 pt] ***c.*** For a similar system with the Hamiltonian $\widehat{H} = \widehat{H}_0 + \frac{1}{2}\hbar\Omega \left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$ expressed in the $\{|1\rangle\,, |2\rangle\}$ basis, specify the appropriate Schrödinger equation and compute its energies exactly.

[7 pt] ***d.*** Apply your results from **a–b** to a linear superposition of two states (detected as particles), initially in the state $|\Psi_{\overline{\alpha}}\rangle$, flying with a relativistically large linear momentum $\mathbf{p}$. Prove that any oscillation in $P_{\alpha\to\overline{\alpha}}(t)$ implies unequal relativistically invariant masses of the two particles, and compute the oscillation frequency to leading order in the (small) masses of the two particles.

Problem 7.

Consider a rotating heteronuclear diatomic molecule constrained to move only in a plane (two dimensions). Assume that the molecule does not undergo translational motion. Indeed, it only has rotational kinetic energy about its center of mass. The quantized energy levels of a diatomic molecule in two dimensions are $\epsilon_J = hcBJ^2$, with $J = 0, 1, 2, 3 \ldots$ and degeneracies $g_J = 2$ for $J \neq 0$ and $g_J = 1$ when $J = 0$. As usual, $B = h/8\pi^2 Ic$, where I is the moment of inertia.

[6 pt] ***a.*** Assuming high temperature ($k_B T \gg hcB$), derive the partition function Z_{rot} for an individual diatomic molecule in two dimensions.

[12 pt] ***b.*** Determine the thermodynamic energy E and heat capacity C_V in the high temperature limit for a set of indistinguishable, independent, heteronuclear diatomic molecules constrained to rotate in a plane.

[7 pt] ***c.*** Compare these results to those for an ordinary diatomic rotor in three dimensions. Comment on the differences and discuss briefly in terms of the number of degrees of freedom required to describe the motion of a diatomic rotor in a plane.