The Howard University
Department of Physics and Astronomy

Master of Science Comprehensive and Doctor of Philosophy Qualifying Exam

Written exam: Classical Physics

August 27, 2013

Attempt to solve ***four*** problems, at least one from each group. Circle the numbers below to indicate the problems you chose to solve:

1 2 3	4 5	6 7
Group A	Group B	Group C

1. Write in your chosen code-letter here: ☐
2. Write your code-letter and a page number (in sequential order) on the *top right-hand* corner of each submitted answer sheet.
3. Write only on one side of your answer sheets.
4. Start each problem on a new answer sheet.
5. Stack your answer sheets by problem and page number and then staple them (at the top *left-hand corner*), with this cover sheet on the top.

Problem 1.

Calculate the gravitational potential and gravitational force field on the axis of:

[6 pt] ***a***. A thin uniform disc of mass M and radius R.

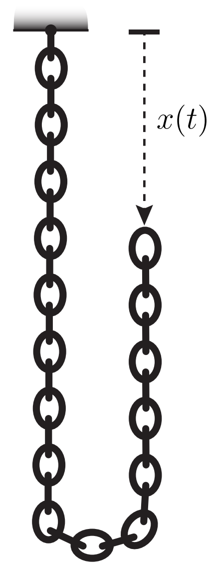
[6 pt] ***b***. A thin uniform annulus of mass M having inner and outer radii R_1 and R_2, respectively.

[6 pt] ***c***. Show that in the limit $z \gg R$, the potential in part ***a*** reduces to $-GM/z$.

[7 pt] ***d***. Show that in the limit $z \ll R$, the force field in part ***a*** reduces to a constant $-2GM\hat{z}/R^2$ (for positive z).

Problem 2.

A uniform flexible chain of length l and mass M is initially suspended with its two ends close together and at the same elevation, and then one end is released. Consider a one dimensional approximation to this two-dimensional problem, in which the chain is represented by two vertical segments connected by a horizontal cross piece which is sufficiently short that its contribution to the kinetic and potential energies may be neglected; see figure to the right.



[6 pt] ***a***. Find the Lagrangian of the chain and obtain the Lagrange equations.

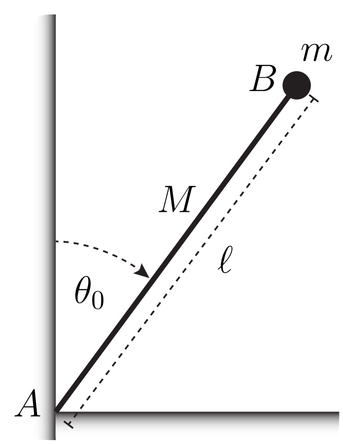
[6 pt] ***b***. Prove that the energy is conserved.

[7 pt] ***c***. Using conservation of energy, find the speed of the falling end as a function of its position, x.

[6 pt] ***d***. Find the tension force at the fixed end as a function of x.

Problem 3.

A thin uniform rod AB of mass M and length ℓ is placed with the end A against the base of a wall and inclined at an angle θ_0 to the vertical; see figure to the right. A point mass m is attached to the end B and the rod is released from rest to fall under the influence of gravity.



[6 pt] ***a***. Find the moment of inertia of the rod with the point mass about the axis of rotation.

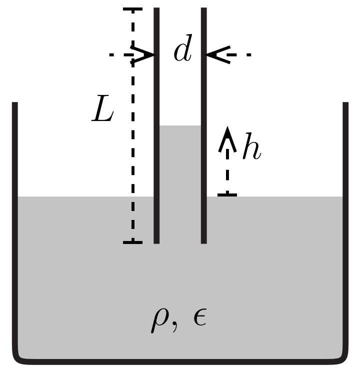
[7 pt] ***b***. Find the Hamiltonian of the system.

[6 pt] ***c***. Find the Hamilton equations for the system.

[6 pt] ***d***. Find the time it takes the rod to reach the ground. (The result can be expressed as quadratures.)

Problem 4.

An $L \times L$ square parallel-plate capacitor and separation d is charged to a potential V and disconnected from the battery, then vertically lowered into a large reservoir of dielectric liquid of relative dielectric constant ϵ, density ρ, and negligible surface tension, until the liquid fills half the space between the plates; see figure to the right.



[6 pt] ***a.*** Calculate the capacitance of the so-constructed capacitor.

[6 pt] ***b.*** Calculate the electric field between the capacitor plates.

[6 pt] ***c.*** Calculate the surface charge density on the plates.

[7 pt] ***d.*** Calculate the height (*vs.* outside level) to which the liquid has risen between the plates above the outside surface level.

Problem 5.

A perfectly conducting sphere of radius R is uniformly charged over its surface to a total charge Q, and is rotating about its axis with a constant angular velocity ω.

[6 pt] ***a.*** Compute the electric current distribution $\vec{j}(\vec{r})$.

[6 pt] ***b.*** Assuming $\vec{\nabla}\cdot\vec{A} = 0$, determine the differential equations and boundary conditions that the vector potential $\vec{A}(\vec{r})$ must satisfy.

[6 pt] ***c.*** Using the cylindrical symmetry of the system, compute $\vec{A}(\vec{r})$ inside and outside the sphere.

[7 pt] ***d.*** Compute $\vec{B} = \vec{\nabla}\times\vec{A}$ inside and outside the rotating charged sphere.

[With cylindrical symmetry, choosing $\vec{A}(\vec{r}) = \hat{e}_\phi\, A_\phi(r,\theta)$, so $(\vec{\nabla}^2\vec{A})_\phi = [\vec{\nabla}^2 - \frac{1}{r^2\sin^2\theta}]A_\phi$ may be useful.]

Problem 6.

A system is composed of N identical subsystems that have energy 0 or ϵ. The total energy of the system is equal to E_0.

[8 pt] ***a.*** Given the thermodynamic limit, in identically equilibrium, find the temperature of the system in terms of the quantities mentioned above.

[9 pt] ***b.*** Find the entropy of a separate, second system, an ideal gas with N_G particles in a volume V at a temperature T_G.

[8 pt] ***c.*** Now, bring both systems into thermal contact. The composite system can be considered isolated. Compute their equilibrium temperature in terms of the quantities given previously.

Problem 7.

Fluctuation Relations: For the Grand Canonical ensemble where κ_T is the isothermal compressibility and k_B is the Boltzman constant,

[12 pt] ***a.*** Prove: $\left\langle \left(N - \langle N \rangle\right)^2 \right\rangle = k_B T n^2 \kappa_T$.

[13 pt] ***b.*** Prove: $\left\langle \left(\mathcal{H} - \mu N - \langle \mathcal{H} - \mu N \rangle\right)^2 \right\rangle = k_B T^2 C_{V,\mu}$.

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August 29, 2013

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Problem 1.

An electron of total energy 1.40 MeV collides with another electron, initially at rest in the lab frame.

[6 pt] ***a.*** Compute the total energy and momentum of the system in the lab frame.

[6 pt] ***b.*** Compute the velocity of the center of mass (CM) in the lab frame.

[6 pt] ***c.*** Compute the total energy of the pair of electrons in the CM frame.

[7 pt] ***d.*** If the target electron scatters 45° from the direction of the incoming electron, compute the direction and magnitude of the momentum of the incoming electron.

The rest energy of the electron is 0.511 MeV.

Problem 2.

Consider the lowest-order relativistic corrections to the linear harmonic oscillator, the non-relativistic Hamiltonian of which is $\boldsymbol{H}_0 = \frac{1}{2M}\boldsymbol{P}^2 + \frac{1}{2}M\omega^2 x^2$.

[6 pt] ***a.*** By expanding the relativistic kinetic energy, obtain the first order relativistic correction to the energy and use this to define the relativistic perturbation operator $\boldsymbol{H}_r$.

[6 pt] ***b.*** Compute the correction to all energy levels to lowest nonzero order in perturbation theory.

[6 pt] ***c.*** Prove the relation $2Mc^2\langle n|\boldsymbol{H}_r|n\rangle = 2E_n^{(0)}\langle n|V|n\rangle - \langle n|V^2|n\rangle - (E_n^{(0)})$.

[7 pt] ***d.*** Compute, independently, the correction to all energy levels to lowest nonzero order in perturbation theory from the anharmonic perturbation $\boldsymbol{H}' = \lambda x^4$, and verify that this result agrees with your results in parts ***b*** and ***c***.

Problem 3.

Two spin-$^1/_2$ electrons are constrained to the 1-dimensional interval $x \in [0, L]$ and arranged to have parallel spins, but otherwise move freely.

[6 pt] ***a.*** Determine the properly normalized 1-electron wave-functions $\psi_i(x)$ for both electrons, $i = 1, 2$.

[6 pt] ***b.*** Using $\psi_i(x)$, write down the most general wave-function for this 2-electron system.

[6 pt] ***c.*** Compute the probability that both electrons are in the same half of the interval.

[7 pt] ***d.*** Compute the probability that the two electrons are in different halves of the interval.

Problem 4.

A particle of mass M is moving in one dimension subject to the potential $V(x) = +\infty$ for $x < 0$ and $V(x) = -V_0\, a\, \delta(x-a)$ for $x > 0$ and with $V_0, a > 0$.

[8 pt] ***a.*** Determine all boundary conditions on the wave-function $\phi(x)$ for this particle.

[8 pt] ***b.*** Determine the general solution of the Schrödingier equation, and all available conditions on the integration constants and the energy.

[9 pt] ***c.*** Determine for which values of V_0, a can there exist a bound state, and compute the critical value of V_0 (with a fixed) for which the ground state energy vanishes.

Problem 5.

A spinless particle of mass M and charge q is restricted to the (x, y)-plane, where it experiences the harmonic potential $\frac{1}{2}M\omega^2(x^2+y^2)$ and the constant perpendicular magnetic field $\vec{B} = B_0\hat{\mathrm{e}}_z$.

[6 pt] ***a.*** Determine the Hamiltonian operator governing the dynamics of this particle.

[6 pt] ***b.*** Prove that the Schrödinger equation may be solved in the polar form, $\psi = N f(\rho)e^{im\phi}$, and determine the differential equation for $f(\rho)$.

[6 pt] ***c.*** Solve this equation exactly for the wave-functions and the energies.

[7 pt] ***d.*** Considering now B_0 as small, compute the lowest order perturbative correction to the energies and compare with the exact result.

[Toggling between Cartesian and polar coordinates may prove useful.]

Problem 6.

Two spin-$^1/_2$ particles are separated by a distance $\vec{a} = a\,\hat{e}_z$ and interact only through the magnetic dipole Hamiltonian $H = \frac{1}{a^3}\,\vec{\mu}_1{\cdot}\vec{\mu}_2 - \frac{3}{a^5}(\vec{\mu}_1{\cdot}\vec{a})(\vec{\mu}_2{\cdot}\vec{a})$, where $\vec{\mu}_i$ is the magnetic moment of the i^{th} particle, assumed to be proportional to its spin, $\vec{S}_i$.

[6 pt] ***a.*** Specify the Hamiltonian operator $\boldsymbol{H}$ in terms of the spin-operators $\vec{\boldsymbol{S}}_i$.

[6 pt] ***b.*** Defining $\vec{\boldsymbol{S}} := \vec{\boldsymbol{S}}_1 + \vec{\boldsymbol{S}}_2$, specify $\boldsymbol{H}$ in terms of the spin-operators $\boldsymbol{S}^2$ and $\boldsymbol{S}_z$.

[6 pt] ***c.*** Compute the energies for all the states of this system.

[7 pt] ***d.*** If an external magnetic field $\vec{B} = B_0\,\hat{e}_z$ is turned on, determine the new Hamiltonian operator $\boldsymbol{H}_{tot} = \boldsymbol{H} + \boldsymbol{H}_{\vec{B}}$, and compute the shifted energies in this system.

Problem 7.

A plane-wave of energy E is incident upon a axially symmetric potential $V = 0$ for $\rho > a$, $V = V_0 > 0$ for $\rho \leqslant a$, and scatters off of it.

[5 pt] ***a.*** Write down the Schrödinger equation, separate variables, and

[5 pt] ***b.*** Determine the general solutions for $E < V_0$ and $E > V_0$, and all boundary conditions.

[5 pt] ***c.*** Derive the exact condition specifying the phase-shift δ_0 in the scattered S-wave.

[5 pt] ***d.*** Determine the behavior of δ_0 when $V_0 \to \infty$.

[5 pt] ***e.*** Determine δ_0 when $E \to 0$, and compute the corresponding total cross-section.

[Recall that for *central* potentials, the substitution $R(r) = \frac{u(r)}{r}$ was useful.]

Problem 8.

There is a model of the thermal behavior of crystalline solids, according to which each of the N atoms of the solid behaves like three independent harmonic oscillators. The $3N$ harmonic oscillators (which are on distinguishable sites) all have the same frequency, ω_0.

[8 pt] ***a.*** Write an expression for the possible energy levels ϵ_n of the n^{th} harmonic oscillator.

[8 pt] ***b.*** Calculate an expression for the Helmholtz free energy, $F(N,T)$, of the solid.

[8 pt] ***c.*** Find the heat capacity, $C(T)$, as a function of the temperature. Give the general formula and, to leading order, show explicitly how to obtain the low-temperature and high-temperature limits. Sketch a plot of C versus T.