The Howard University
Department of Physics and Astronomy

Master of Science Comprehensive and Doctor of Philosophy Qualifying Exam

Written exam: Classical Physics

August 19, 2014

Attempt to solve ***four*** problems, at least one from each group. Circle the numbers below to indicate the problems you chose to solve:

1 2	3 4	5 6 7
Group A	Group B	Group C

1. Write in your chosen code-letter here: ☐
2. Write your code-letter and a page number (in sequential order) on the *top right-hand* corner of each submitted answer sheet.
3. Write only on one side of your answer sheets.
4. Start each problem on a new answer sheet.
5. Stack your answer sheets by problem and page number and then staple them (at the top *left-hand corner*), with this cover sheet on the top.

Problem 1.

If the solar system were immersed in a uniformly dense spherical cloud of weakly-interacting massive particles (WIMPs), then objects in the solar system would experience gravitational forces from both the sun and the cloud of WIMPs such that

$$F_r = -\frac{k}{r^2} - br,$$

where r is the radial distance to the center, k and b are constants.

[7 pt] ***a.*** Determine the Lagrangian of a particle of mass m in this force field.

[6 pt] ***b.*** Determine the corresponding Hamiltonian.

[6 pt] ***c.*** Derive the equations of motion.

[6 pt] ***d.*** Compute the frequency of small radial oscillations for a nearly circular orbit. Assume that near the circular orbit, the extra force due to the WIMPs is very small, *i.e.*, $b \ll k/r^3$.

Problem 2.

A particle with mass m and charge e is moving in the (x, y)-plane in a magnetic field described by the vector potential $\vec{A} = \frac{1}{2}B(x\,\hat{\mathbf{e}}_y - y\,\hat{\mathbf{e}}_x)$, where B is the magnitude of the magnetic field (uniform).

[6 pt] ***a.*** Determine the Hamiltonian, H.

[5 pt] ***b.*** Consider the following transformation of coordinates:

$$x = \frac{1}{\sqrt{m\omega}}\big(\sqrt{2P_1}\sin Q_1 + P_2\big), \qquad y = \frac{1}{\sqrt{m\omega}}\big(\sqrt{2P_1}\cos Q_1 + Q_2\big), \tag{1}$$

$$P_x = \frac{\sqrt{m\omega}}{2}\big(\sqrt{2P_1}\cos Q_1 - Q_2\big), \qquad P_y = \frac{\sqrt{m\omega}}{2}\big(-\sqrt{2P_1}\sin Q_1 + P_2\big). \tag{2}$$

Prove that the Poisson bracket $[x, P_x]_{Q,P} = 1$.

[2 pt] ***c.*** Is this the only necessary condition for the transformation to be canonical? If not, state the complete set of conditions.

[6 pt] ***d.*** Rewrite the hamiltonian H in terms of the set of coordinates $\{Q_1, Q_2, P_1, P_2\}$, with $\omega = \frac{eB}{mc}$.

[6 pt] ***e.*** Derive Hamilton's equations, and solve them. Interpret the result.

Problem 3.

The space between two concentric spheres of radii R_1 and R_2 ($R_1 < R_2$) is charged to a volume charge density given by $\rho = \alpha/r^2$ (α is a constant).

[5 pt] ***a.*** Compute the total charge, q.

[6 pt] ***b.*** Compute the electric potential, ϕ in all the space.

[8 pt] ***c.*** Compute the electric field strength, $\vec{E}$ in all the space.

[6 pt] ***d.*** Determine the limiting case $R_2 \to R_1$, assuming q to be constant. Explain the result.

Problem 4.

Consider the propagation of electromagnetic fields in a non-conducting medium with constant permeability and susceptibility.

[6 pt] ***a.*** Write down Maxwell's equations in this medium.

[6 pt] ***b.*** Show that if $\rho = \vec{\jmath} = 0$, $\vec{E}$ and $\vec{B}$ satisfy the wave equation. Find an expression for the wave velocity.

[6 pt] ***c.*** Write down the plane wave solutions for $\vec{E}$ and $\vec{B}$ and show how $\vec{E}$ and $\vec{B}$ are related.

[7 pt] ***d.*** Discuss the reflection and refraction of the electromagnetic waves at a plane interface between two dielectrics and derive the relationships between the angles of incidence, reflection and refraction.

Problem 5.

Derive the following relations between the thermodynamic variables: temperature (T), volume (V), pressure (P), entropy (S) and internal energy (U):

[5 pt] ***a.*** $(\frac{\partial T}{\partial V})_S = -(\frac{\partial P}{\partial S})_V$.

[5 pt] ***b.*** $(\frac{\partial T}{\partial P})_S = (\frac{\partial V}{\partial S})_P$.

[5 pt] ***c.*** $(\frac{\partial P}{\partial T})_V = (\frac{\partial S}{\partial V})_T$.

[5 pt] ***d.*** $(\frac{\partial V}{\partial T})_P = -(\frac{\partial S}{\partial P})_T$.

[5 pt] ***e.*** $(\frac{\partial U}{\partial V})_T = T(\frac{\partial P}{\partial T})_V - P$.

Problem 6.

An assembly of N particles of spin 1/2 are lined up on a straight line. Only nearest neighbors interact. When the spins of both neighbors are both up or both down, their interaction energy is J. When one spin is up and the other is down, the interaction energy is $-J$.

[7 pt] ***a.*** Determine the partition function Z of the assembly at temperature T.

[6 pt] ***b.*** Compute the free energy of the system, F.

[6 pt] ***c.*** Compute the internal energy, U.

[6 pt] ***d.*** Compute the entropy, S.

Problem 7.

Consider an adsorbent surface having n sites, each of which can adsorb one gas molecule. This surface is in contact with a vapor with chemical potential μ (determined by the pressure P and temperature T). Assume that the adsorbed molecule is monoatomic and has energy ϵ_0 compared to one in a free state.

[6 pt] ***a.*** If N molecules are adsorbed, compute the number of possible different configurations of the system.

[7 pt] ***b.*** Compute the Grand Canonical Partition Function of the system.

[6 pt] ***c.*** Calculate the coverage ratio θ, *i.e.*, the ratio of adsorbed molecules to adsorbing sites on the surface.

[6 pt] ***d.*** Compute $\theta(P, T)$ assuming that the vapor is an ideal gas.

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Problem 1.

A pion decays into a muon and an antineutrino, $\pi^- \to \mu^- + \overline{\nu}_\mu$. All three are particles massive, although $m_\nu \ll m_\mu < m_\pi$.

[5 pt] ***a.*** Working within the CM (center-of-momenta) coordinate frame, compute the total (relativistic) energy of the emitted muon in terms of only the masses and universal constants.

[2+5 pt] ***b.*** Compute the total (relativistic) energy of the emitted neutrino in terms of only the masses and universal constants. Compute the ratio of E_ν/E_μ to first order in $\left(\frac{m_\nu^2}{m_\pi^2}\right) \ll 1$.

[8 pt] ***c.*** Compute the magnitude of the (relativistic) linear momenta of the emitted muon and antineutrino in terms of only the masses and universal constants.

[5 pt] ***d.*** In the limit $m_\nu \to 0$, compute the muon's energy and magnitude of its linear momentum.

Problem 2.

A particle of mass m and charge q enters a region with the vertically upward oriented homogeneous magnetic field $\vec{B} = B\hat{e}_z$ with the horizontal velocity $\vec{v} = v\hat{e}_x$.

[5 pt] ***a.*** Determine the radius of curvature of the trajectory of this particle while traveling through this magnetic field, calculating separately for the non-relativistic and the relativistic regime.

[5 pt] ***b.*** Assuming that the region with the magnetic field is sufficiently broad, will the motion of the particle be periodic? If so, calculate its period, frequency and angular momentum magnitude.

[7 pt] ***c.*** For particles traveling in closed orbits within this magnetic field, use wave-particle duality to determine the condition that quantizes their energy, assuming non-relativistic motion. Compute the quantized energy and angular momentum spectra.

[8 pt] ***d.*** For an electron, compute the range of the intensity of the homogeneous magnetic field so that the speed of the electron orbiting would not exceed the speed of light.

Problem 3.

A particle of mass m and electric charge q moves in a horizontal plane ($z=0$), within a constant magnetic field $\vec{B} = B\hat{e}_z$ but otherwise freely. Use that the physical linear momentum of this charged particle is modified to $\vec{\boldsymbol{P}} = \frac{\hbar}{i}\vec{\nabla} - q\vec{A}$ and $\vec{A} = \frac{1}{2}(\vec{B}\times\vec{r})$.

[(2+5) pt] ***a.*** Compute $\vec{A}$, verify that $(\vec{\nabla}\cdot\vec{A}) = 0$ and show that the non-relativistic Hamiltonian is $\boldsymbol{H} = -\frac{\hbar^2}{2m}\vec{\nabla}^2 - \frac{qB}{2m}\boldsymbol{L}_z + \frac{(qB)^2}{8m}\rho^2$, where $\rho = \sqrt{x^2+y^2}$ the "radial" coordinate in the cylindrical coordinate system. and $\boldsymbol{L}_z$ is the z-component of the angular momentum operator

[5 pt] ***b.*** For the stationary state function $\Psi(\rho,\phi,z=0,t) = e^{i\omega t}R(\rho)e^{i\mu\phi}$ to satisfy the Schrödinger equation, determine the differential equation that $R(\rho)$ must satisfy.

[6 pt] ***c.*** For the radial function $R(\rho) = e^{-a\rho^2}f(\rho)$, determine α so that the no-derivative terms in the differential equation that $f(\rho)$ satisfies (the "effective potential") would contain only two different powers of ρ. Determine the resulting differential equation that $f(\rho)$ satisfies.

[8 pt] ***d.*** Writing $f(\rho) = \sum_{k=0}^{\infty} c_k\rho^{k+s}$, compute the recursion relation for c_k so that $R(\rho) = e^{-\alpha\rho^2}f(\rho)$ is the radial factor in the stationary state function, show that this series must be terminated (why?), show that this quantizes the energy, and find the energy spectrum.

Problem 4.

Consider a modified harmonic oscillator, with the Hamiltonian $\boldsymbol{H} = \hbar\omega(\boldsymbol{a}^\dagger\boldsymbol{a}+\frac{1}{2}) + A\boldsymbol{a}\boldsymbol{a} + A^*\boldsymbol{a}^\dagger\boldsymbol{a}^\dagger$, where ω is the characteristic frequency of the linear harmonic oscillator and A is a suitable complex constant. Write $\boldsymbol{H}_0 = \hbar\omega(\boldsymbol{a}^\dagger\boldsymbol{a}+\frac{1}{2})$ and $\boldsymbol{H}_0\,|n\rangle = \hbar\omega(n+\frac{1}{2})\,|n\rangle$, as usual.

[3 pt] ***a.*** Treating $\boldsymbol{H}_1 = A\boldsymbol{a}\boldsymbol{a} + A^*\boldsymbol{a}^\dagger\boldsymbol{a}^\dagger$ as a small perturbation, compute the first order (stationary state) perturbative corrections to the energy.

[6 pt] ***b.*** Still treating $\boldsymbol{H}_1$ as a small perturbation, compute the first order (stationary state) perturbative corrections to the stationary state $|n\rangle$.

[6 pt] ***c.*** Still treating $\boldsymbol{H}_1$ as a small perturbation, compute the second order (stationary state) perturbative corrections to the energy.

[7 pt] ***d.*** Determine the eigenstate $\boldsymbol{H}\,|E\rangle = E\,|E\rangle$ of the full Hamiltonian as a formal expansion $|E\rangle = \sum_{n=0}^{\infty} c_n\,|n\rangle$; determine all the coefficients c_n via the complete sequence of recursion relations.

[3 pt] ***e.*** Determine the degeneracy of the eigenstates $\boldsymbol{H}\,|E\rangle = E\,|E\rangle$.

Problem 5.

A spin-½ nucleus is placed in a large constant magnetic field $\vec{B}_0 = B_0\,\hat{\mathbf{e}}_z$, with a weaker oscillating magnetic field

$$\vec{B}_1 = B_1\cos(\omega t)\,\hat{\mathbf{e}}_x + B_1\sin(\omega t)\,\hat{\mathbf{e}}_y, \qquad B_1 < B_0,$$

rotates in the (x, y)-plane. Denote $\omega_0 \stackrel{\text{def}}{=} \mu B_0/\hbar$ and $\omega_1 \stackrel{\text{def}}{=} \mu B_1/\hbar$, where μ is the magnitude of the magnetic dipole moment of the nucleus. Use the interaction picture, where the Hamiltonian reduces to $\boldsymbol{H} = \mu\,\vec{B}\cdot\vec{\boldsymbol{\sigma}}$, and where $\vec{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3)$ is the vector of 2×2 Pauli matrices.

[5 pt] ***a.*** Using the 2-component matrix notation, $\Psi(t) = \left[\begin{smallmatrix}\psi_1(t)\\ \psi_2(t)\end{smallmatrix}\right]$, derive the system of 1st-order differential coupled equations for $\psi_i(t)$ implied by the Schrödinger equation.

[5 pt] ***b.*** Show that the system of two 1st-order differential equations from part **a** is equivalent to two uncoupled 2nd-order differential equations for each of $\psi_i(t)$ separately, and compute their general solutions, exhibiting four integration constants.

[5 pt] ***c.*** Using the original Schrödinger equation, reduce the number of independent integration constants from part **b** to two, *i.e.*, compute the general solution of the original Schrödinger equation in 2-component matrix form.

[3 pt] ***d.*** If the nucleus was in the "spin-up" state at $t = 0$, $\Psi(0) = \left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right]$, determine the state of the nucleus at all later times, $t > 0$.

[7 pt] ***e.*** Calculate the probability that the nucleus will be in the "spin-down" state at a particular, fixed later time, $T > 0$.

Problem 6.

Consider a single hydrogen atom, initially in the $|2,0,0\rangle$ state. Neglect the motion of the proton, the magnetic dipole moments (and spins) of the electron and the proton and relativistic effects, and restrict your analysis to $n=2$ states.

[3 pt] ***a.*** In this approximation and knowing that the energy of the ground state is $-13.6\,\text{eV}$, list the energies of the $n=2$ states.

[7 pt] ***b.*** If the hydrogen atom was placed in a weak electric field $\vec{E} = \mathcal{E}\,\hat{\mathbf{e}}_z$ for a long time, compute the energy levels of the $n=2$ states to lowest order in the magnitude of the electric field, $\mathcal{E}$.

[10 pt] ***c.*** If the weak electric field was only turned on at the time $t=0$ when the hydrogen atom was known to be in the $|2,0,0\rangle$ state, and made to alternate as $\vec{E} = \mathcal{E}\cos(\Omega t)\,\hat{\mathbf{e}}_z$ for all $t > 0$, compute the probability that the hydrogen atom is found in the $|2,1,0\rangle$ state after a long enough time $T \gg \frac{1}{\Omega}$ and to lowest order in the magnitude of the electric field, $\mathcal{E}$.

[5 pt] ***d.*** Compute the frequency with which the probability that the hydrogen atom is at a later time $T > 0$ found in the $|2,1,0\rangle$ state reaches its maximum.

Problem 7.

Consider a photon gas enclosed in a volume V and in equilibrium at temperature T. The photon is a massless particle, so that its energy is relativistic and given by $\epsilon = pc$.

[6 pt] ***a.*** Determine the chemical potential of the gas. Explain your answer.

[7 pt] ***b.*** Determine how the number of photons in the volume depends upon the temperature.

[6 pt] ***c.*** Determine the form of the spectral energy density $\rho(\omega)$.

[6 pt] ***d.*** Determine the temperature dependence of the average energy $\langle E \rangle$.

Possibly useful information:

Symbol	**Value**	**Physical Meaning**
c	$3.00\times10^{8}\,\mathrm{m/s}$	speed of light in vacuum
$\hbar$	$1.05\times10^{-34}\,\mathrm{J\,s}$	Planck's constant
m_e	$9.11\times10^{-31}\,\mathrm{kg}$	electron mass
e	$1.60\times10^{-19}\,\mathrm{C}$	unit charge
a	$5.29\times10^{-11}\,\mathrm{m}$	Bohr radius

$$\vec\nabla\cdot(\vec\nabla f)=\vec\nabla^2 f,\qquad \vec\nabla\times(\vec\nabla\times\vec X)=\vec\nabla(\vec\nabla\cdot\vec X)-\vec\nabla^2\vec X.$$

In cylindrical coordinates,

$$\vec\nabla\cdot\vec X=\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho}+\frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z},\qquad \vec\nabla\times\vec X=\frac{1}{\rho}\begin{vmatrix}\hat{\mathbf{e}}_\rho & \rho\hat{\mathbf{e}}_\phi & \hat{\mathbf{e}}_z\\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ X_\rho & \rho X_\phi & X_z\end{vmatrix}$$

and

$$\vec\nabla^2 X=\left[\frac{\partial^2 X}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial X}{\partial\rho}\right]+\frac{1}{\rho^2}\frac{\partial^2 X}{\partial\phi^2}+\frac{\partial^2 X}{\partial z^2}.$$

In spherical coordinates,

$$\vec\nabla\cdot\vec X=\frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi},\qquad \vec\nabla\times\vec X=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & r\sin\theta\,\hat{\mathbf{e}}_\phi\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ X_r & rX_\theta & r\sin\theta X_\phi\end{vmatrix}$$

and

$$\vec\nabla^2 X=\frac{\partial^2 X}{\partial r^2}+\frac{2}{r}\frac{\partial X}{\partial r}+\frac{1}{r^2}\frac{\partial^2 X}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial X}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 X}{\partial\phi^2}.$$

The hydrogen-like atom wave-functions neglecting the electron and the proton spin:

$$|n,\ell,m\rangle=\sqrt{\Big(\frac{2Z}{na}\Big)^3\frac{(n-\ell-1)!}{2n(n+\ell)!}}\;e^{-\varrho/2}\,\varrho^\ell\,L^{2\ell+1}_{n-\ell-1}(\varrho)\,Y^m_\ell(\theta,\phi),\qquad \varrho\overset{\text{def}}{=}\frac{2Z}{na}r,$$

where $L^k_n(\varrho)=\dfrac{e^\varrho\varrho^{-k}}{n!}\dfrac{\mathrm{d}^n(e^{-\varrho}\varrho^{n+k})}{\mathrm{d}\varrho^n}$ are the associated Laguerre polynomials, and

$$Y^0_0(\theta,\phi)=\frac{1}{\sqrt{4\pi}},\qquad Y^0_1(\theta,\phi)=-\sqrt{\frac{3}{4\pi}}\cos\theta,\qquad Y^0_1(\theta,\phi)=\sqrt{\frac{3}{8\pi}}\sin\theta\,e^{i\phi},$$

$$Y^0_2(\theta,\phi)=\sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta-\frac{1}{2}\Big),\qquad Y^1_2(\theta,\phi)=-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta,\qquad Y^2_2(\theta,\phi)=\sqrt{\frac{15}{32\pi}}\sin^2\theta\,e^{2i\phi},\ldots$$

are the first few spherical harmonics (in the Condon-Shotley convention).