The Howard University
Department of Physics and Astronomy

Master of Science Comprehensive and Doctor of Philosophy Qualifying Exam

Written exam: Classical Physics

August 25, 2015

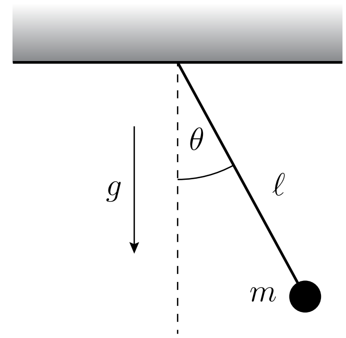
Attempt to solve ***four*** problems, at least one from each group. Circle the numbers below to indicate the problems you chose to solve:

1 2	3 4 5	6 7
Group A	Group B	Group C

1. Write in your chosen code-letter here:
2. Write your code-letter and a page number (in sequential order) on the *top right-hand* corner of each submitted answer sheet.
3. Write only on one side of your answer sheets.
4. Start each problem on a new answer sheet.
5. Stack your answer sheets by problem and page number and then staple them (at the top *left-hand corner*), with this cover sheet on the top.

Problem 1.

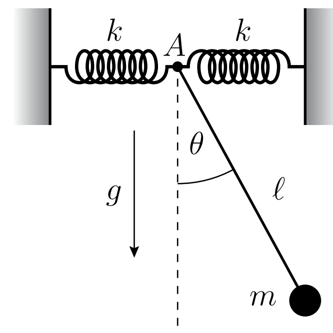
A simple pendulum is shown in the top figure to the right:



[5 pt] ***a.*** Show that the angular frequency for the simple harmonic motion is $\omega = \sqrt{\frac{g}{l}}$.

[5 pt] ***b.*** Write the period of the pendulum for small oscillations.

Consider now the system displayed in the lower figure to the right. The point A is free to move along a horizontal line under the action of the springs having a constant k. After displacing point A by a small distance x to the right, do the following:



[5 pt] ***c.*** Set up the Lagrangian for this system.

[5 pt] ***d.*** Write the Equations of Lagrange for this system.

[5 pt] ***e.*** Show that for small oscillations, the period T of this pendulum is now $T = 2\pi\sqrt{\frac{mg+2kl}{2kg}}$.

Problem 2.

Consider a homogeneous cube of density ρ, mass M and side of length b.

[6 pt] ***a.*** Calculate the inertia tensor of the cube in a coordinate system with the origin at the center of mass. Clearly indicate the coordinate system in a figure.

[6 pt] ***b.*** Find the principal moments of inertia and principal axes, keeping the origin at the center of mass.

[6 pt] ***c.*** Calculate the inertia tensor in a coordinate system with the origin at one of the corners of the cube. Clearly indicate the coordinate system in a figure.

[7 pt] ***d.*** Find the principal moments of inertia and principal axes, keeping the origin at the corner of the cube.

Problem 3.

A charged particle is moving in a constant magnetic field H, with vector potential $A_y = xH$, $A_z = A_x = 0$.

[5 pt] ***a.*** Write the Hamiltonian.

[5 pt] ***b.*** What are the constants of the motion? Justify.

[5 pt] ***c.*** Obtain the Hamilton equations of motion.

[5 pt] ***d.*** Solve the equations of motion.

[5 pt] ***e.*** Describe the motion of the particle.

Problem 4.

[5 pt] ***a.*** Write the four Maxwell's Equations in differential form and give a short explanation for each equation.

[5 pt] ***b.*** Write the continuity equation.

[5 pt] ***c.*** Derive the wave equation for electromagnetic fields propagating in free space. Begin your derivation with the Ampere-Maxwell equation.

Now, consider the interface between two different nonconducting media.

[5 pt] ***d.*** State the continuity conditions for the field strengths (E, B, D and H) and sketch their derivations, i.e., physical origins.

[5 pt] ***e.*** From these continuity conditions, derive the laws of reflection and refraction.

Problem 5.

A parallel-plate capacitor of plate area $0.2\,\mathrm{m}^2$ and plate spacing $1\,\mathrm{cm}$ is charged to $1{,}000\,\mathrm{V}$ and is then disconnected from the battery. (Use $\epsilon_0 = 8.9 \times 10^{-12}\,\mathrm{C}^2/(\mathrm{N.m}^2)$)

[7 pt] ***a.*** Calculate the capacitance of the capacitor.

[6 pt] ***b.*** Compute the energy stored in the capacitor.

[6 pt] ***c.*** Compute the work required to pull the plates apart to double the plate spacing.

[6 pt] ***d.*** Compute the final voltage on the capacitor.

Problem 6.

[5 pt] ***a.*** Define or state (be brief, but precise and define symbols and/or terms you use):

i. the Zeroth Law of Thermodynamics;
ii. temperature;
iii. absolute temperature;
iv. the First Law of Thermodynamics;
v. the thermodynamic entropy;
vi. the most general statement of the Second Law of Thermodynamics;
vii. the Third Law of Thermodynamics.

Consider a lattice of N atoms, each of which can be in one of two states, of energy 0 or ϵ, respectively. Suppose that the total energy is E. (You may assume that E is an integer multiple of ϵ.)

[7 pt] ***b.*** Using Stirling's approximation: $\ln n! \approx (n \log n - n)\sqrt{2\pi n}$, show that

$$\ln\big(\Omega(N,E)\big) = N\ln N - (E/\epsilon)\ln\big(E/\epsilon\big) - (N{-}E/\epsilon)\ln\big(N - E/\epsilon\big).$$

[6 pt] ***c.*** Compute the temperature $T(N,E)$.

[6 pt] ***d.*** Compute the heat capacity $C(N,E)$.

[6 pt] ***e.*** Compute the Helmholtz Free Energy $A(N,E)$.

Problem 7.

Consider N fixed non-interacting magnetic moments each of magnitude μ_0. The system is in thermal equilibrium at temperature T and is in a uniform external magnetic field B. Each magnetic moment can be oriented parallel or antiparallel to B. Calculate the following:

[7 pt] ***a.*** the partition function,

[6 pt] ***b.*** the specific heat,

[6 pt] ***c.*** the thermal average magnetic moment.

[6 pt] ***d.*** Show that in the high temperature limit the Curie Law is satisfied, i.e. the magnetic susceptibility χ is proportional to $1/T$.

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Problem 1.

Observers on Earth see two meteorites flying on a direct collision course towards each other, with equal speeds of 0.60 c. At Earth-time $t_E = 0.00\,\text{s}$, the two meteorites are measured (in Earh-frame) to be $\ell = 900,000\,\text{km}$ apart from each other.

[5 pt] ***a.*** Compute the Earth-time at which the two meteorites will collide.

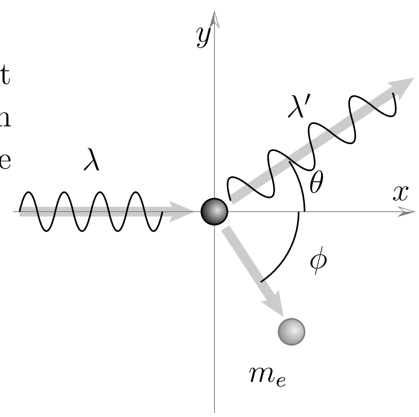
[5 pt] ***b.*** Compute the speed with which one meteorite is approaching the other, from the rest-frame of the latter meteorite.

[5 pt] ***c.*** Compute the time that elapses in the meteorite frame between the initial moment $t_M = 0.00 = t_E$ and the collision.

[10 pt] ***d.*** If both meteorites had the same mass, m, before the collision and they clump together without ejecting any debris after the collision, compute the mass of the so-fused object and its relativistic momentum in Earth-frame.

Problem 2.

An X-ray photon of wave-length λ collides with an electron that was initially at rest. The photon deflects from its initial direction by an angle θ, while the electron recoils at an angle ϕ from the photon's initial direction.



[5 pt] ***a.*** Using the coordinate system and symbols given in the figure to the right, express the energy and linear momentum of both the photon and the electron, before and after the collision.

[6 pt] ***b.*** Write down the conservation equations which completely determine all parameters in this scattering.

[6 pt] ***c.*** Derive Compton's formula, $\lambda' - \lambda = \frac{h}{m_e\, c}(1 - \cos\theta)$ from the conservation equations you specified in the previous part.

[8 pt] ***d.*** An incoming neutrino ($m_\nu \approx 0$) of energy E_ν is absorbed by a neutron, which emits an electron at the angle θ and thus turns into a proton. Using the same conservation laws, that $m_n \approx m_p + m_e$ and neglecting the proton recoil, show that $|\vec{p}_e| = \sqrt{E_\nu(E_\nu + 2m_e c^2)}/c$.

Problem 3.

A particle of mass m is confined to move in one dimension, and under the influence of a potential

$$V(x) = \infty \quad \text{for } x < 0, \qquad V(x) = -V_0 \quad \text{for } 0 < x < a, \qquad V(x) = 0 \quad \text{for } x > a.$$

[5 pt] ***a.*** Sketch the situation and specify all the boundary conditions on the wave-function representing this particle.

[5 pt] ***b.*** Specify: the Hamiltonian for this particle, the functional form of its wave-function, and use the boundary conditions to find the equation which determines the bound-state energies.

[5 pt] ***c.*** From your solution so far, determine the minimal value of $V_0 > 0$ so that there would exist at least one bound state.

[5 pt] ***d.*** For scattering states ($E > 0$), derive the condition that relates the energy of the state to its phase-shift between the "inside" ($0 < x < a$) and "outside" ($x > a$) oscillatory solution.

[5 pt] ***e.*** Compute this phase-shift as a function of the energy of the state.

Problem 4.

A particle of mass m is contained within a 1-dimensional box, limiting $x \in [-\frac{L}{2}, \frac{L}{2}]$.

[5 pt] ***a.*** Write down the Schrödinger equation for the wave-function of this particle, specify all the boundary conditions, write down the wave-functions for the complete set of stationary bound states and state their energies.

[5 pt] ***b.*** If the original system is prepared in its ground state and the walls of the box are instantaneously moved symmetrically to twice the width of the box, compute the probability that the system will be found in the ground state of this wider-boxed particle.

[5 pt] ***c.*** If the original system is prepared in its 1st excited state and the walls of the box are again instantaneously moved symmetrically to twice the width of the box, compute the probability that the system will be found in the ground state of this wider-boxed particle.

[5 pt] ***d.*** If the original system is modified by adding to the potential a perturbation $\hat{H}' = \lambda\delta(x)$, compute the lowest-order non-vanishing perturbative corrections to the stationary state energies.

[5 pt] ***e.*** If the perturbation is made oscillatory in time $\hat{H}' \to \hat{H}'(t) = \lambda\delta(x)\cos(\omega t)$, compute the probability that a system initially (at $t = 0$) in the ground state could after a long time $T \gg \omega^{-1}$ be found in its first excited state.

Problem 5.

Consider the linear harmonic oscillator with characteristic frequency ω. Define $\hat{a} := \sqrt{\frac{m\omega}{2\hbar}}\big(\hat{x} + \frac{i}{m\omega}\hat{p}\big)$ and its Hermitian conjugate $\hat{a}^\dagger$ as usual in terms of the position $\hat{x}$ and linear momentum $\hat{p}$. Use the standard basis, $\mathscr{H} = \big\{\,|n\rangle\,,\ n = 0, 1, 2, \cdots\big\}$, where $\hat{a}\,|n\rangle = \sqrt{n}\,|n{-}1\rangle$ and $\hat{a}^\dagger\,|n\rangle = \sqrt{n{+}1}\,|n{+}1\rangle$.

[5 pt] ***a.*** Derive the commutation relation $[\,\hat{a}\,,\,\hat{a}^\dagger\,]$ from the canonical relation, $[\,\hat{x}\,,\,\hat{p}\,] = i\hbar$.

[5 pt] ***b.*** Construct an eigenstate, $|z\rangle$, of the annihilation operator: $\hat{a}\,|z\rangle := z\,|z\rangle$ as a formal expansion $|z\rangle := \sum_n c_n\,|n\rangle$, determining as many of the constants c_n as you can in terms of the eigenvalue, z.

[5 pt] ***c.*** Re-expressing your result as $|z\rangle = \hat{Z}_z\,|0\rangle$, determine the correctly normalized operator $\hat{Z}_z$, and all the values that the eigenvalue z permitted so $|z\rangle$ is normalizable.

[5 pt] ***d.*** Calculate $\hat{U}^{-1}(t)\,\hat{a}\,\hat{U}(t)$, where $\hat{U}(t) := e^{-i\frac{t}{\hbar}\hat{H}}$ is the evolution operator.

[5 pt] ***e.*** Calculate $\hat{U}(t)\,|z\rangle$. Determine from this a physical interpretation of z, and of $|z\rangle$.

***Hint**: Be careful with the limits on $|n\rangle$-summations. You may need the Baker-Campbell-Hausdorff formulae, $e^A e^B = e^B e^A e^{[A,B]}$ and $e^A e^B = e^{A+B-\frac{1}{2}[A,B]}$, both valid if $[A,[A,B]] = 0 = [B,[A,B]]$, and the generally valid $e^{\alpha A}\,B\,e^{-\alpha A} = B + \sum_{k=1}^{\infty}\frac{\alpha^k}{k!}\big[\underbrace{A,[A,\cdots[A}_{k\ \text{times}},\,B\,]\cdots]\big]$.*

Problem 6.

Consider a non-interacting non-relativistic gas of N spin-½ fermions, at temperature $T = 0$ and confined to an $L \times L$ square-shaped surface.

[5 pt] ***a.*** Determine the number of quantum states available to these fermions with linear momentum between p and $p + \mathrm{d}p$.

[8 pt] ***b.*** Observing Pauli's exclusion principle, determine the Fermi energy, E_F, below which all states are filled and above which all states are empty.

[6 pt] ***c.*** Show that the total energy of this ensemble is $E = \frac{1}{2}NE_F$, and compute the average energy.

[6 pt] ***d.*** Using that the volume of the system is $L \times L \times d$ where the thickness is $d \ll L$ and working "per unit thickness," compute the pressure in this ensemble.

Problem 7.

Consider a 1-dimensional harmonic oscillator of mass m and characteristic frequency ω.

[5 pt] ***a.*** Write down the Hamiltonian and its eigenvalues, corresponding to the standard stationary states $|n\rangle$.

[8 pt] ***b.*** Compute the lowest-order relativistic correction to the Hamiltonian in operator form, and compute the lowest-order perturbative non-vanishing corrections to the stationary state energies.

[7 pt] ***c.*** Compute the lowest-order perturbative non-vanishing corrections to the stationary state energies due to the anharmonic perturbation $\hat{H}' = \lambda\hat{x}^4$, and determine the value of λ for which this equals the corrections you found in the previous computation.

[5 pt] ***d.*** Determine if it is possible to choose λ so that the two previously computed lowest-order corrections would cancel, and if so estimate the next-order non-vanishing correction if any.