The Howard University
Department of Physics and Astronomy

Master of Science Comprehensive and Doctor of Philosophy Qualifying Exam

Written exam: Classical Physics

August 23, 2016

Attempt to solve ***four*** problems, at least one from each group. Circle the numbers below to indicate the problems you chose to solve:

1 2	3 4 5	6 7
Group A	Group B	Group C

1. Write in your chosen code-letter here: ☐
2. Write your code-letter and a page number (in sequential order) on the *top right-hand* corner of each submitted answer sheet.
3. Write only on one side of your answer sheets.
4. Start each problem on a new answer sheet.
5. Stack your answer sheets by problem and page number and then staple them (at the top *left-hand corner*), with this cover sheet on the top.

Problem 1.

A spherical pendulum is set in motion as shown in the diagram below:

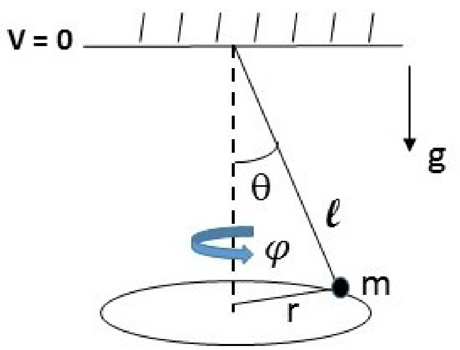
[5 pt] ***a.*** Determine the kinetic energy and potential energy for the system.

[5 pt] ***b.*** Construct the Lagrangian function.

[5 pt] ***c.*** Find the Euler-Lagrange equation of motion.

[5 pt] ***d.*** φ is a cyclical coordinate for this system. Calculate an expression for the conserved quantity of angular momentum, L_z, and derive an analytical expression for g/l at equilibrium (i.e. let $\theta = \theta_0$).

[5 pt] ***e.*** Find the frequency of oscillations, ω, for small displacements about the equilibrium angle, θ_0.



Problem 2.

[8 pt] ***a.*** Prove that the Hamiltonian, H, in most cases represents the total energy of a system and explain under what conditions it is conserved.

[8 pt] ***b.*** Write the Hamiltonian for a simple spring-mass harmonic oscillator and from it obtain its equations of motion.

[9 pt] ***c.*** Use the following Lagrangian:

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{\kappa}{r}$$

to obtain the Hamiltonian corresponding to a planet in orbit around the sun. Here $\kappa = Gmm_s$ and m is the mass of the planet, m_s is the mass of the Sun and G is the universal gravitational constant.

Problem 3.

[5 pt] ***a.*** Write the expression, in MKS units, that is used to calculate the electric potential Φ due to a continuous charge at a point P due to individual elements of charge dq where r is the distance between P and dq.

Line of Charge:

A thin non-conducting rod of length L has a positive charge of uniform linear density λ.

[5 pt] ***b.*** Calculate the electric potential Φ due to the rod at point P, a perpendicular distance d from the left end of the rod. (Helpful integral: $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}|$).

Ring of Charge:

It can be shown that the general form of the solution to Laplace's equation, $\nabla^2\Phi = 0$, for an axially symmetric situation is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-(l+1)})P_l(\cos\theta)$$

where the first few Legendre polynomials are

$$P_0(\cos\theta) = 1; P_1(\cos\theta) = \cos\theta; P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1); P_2(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$$

With the information given above, now consider a circular ring of radius a which has a constant linear charge density λ on its circumference. Take the z-axis to be the axis of symmetry.

[5 pt] ***c.*** Calculate the electric potential on the z-axis (for the Ring of Charge),

[5 pt] ***d.*** Calculate the electric field on the z-axis (for the Ring of Charge).

[5 pt] ***e.*** Calculate the potential for all space (i.e., off axis) for $z < a$ and $z > a$ (for the Ring of Charge). Express your answers as a series in the $P_l(\cos\theta)$.

Problem 4.

A rod of uniform conductance σ, mass density ρ and cross-section A slides with velocity $v_0\,\hat{e}_y$ on a pair of frictionless and perfectly conducting rails located a distance $\frac{1}{2}L$ on both sides (in the x-direction) along the positive y-axis. The rails are connected at $y = 0$ by a perfect conductor, and a constant magnetic field $B_0\,\hat{e}_z$ passes through the area between the rails for all $y > 0$.

[5 pt] ***a.*** Compute the electromotive force in the conducting sliding rod.

[5 pt] ***b.*** Compute the induced current passing through the sliding rod.

[5 pt] ***c.*** Compute the resulting force acting on the sliding rod.

[5 pt] ***d.*** Using Newton's 2nd law, compute obtain a differential equation for the speed of sliding of the rod, and show that $v = v_0\,e^{-t/\tau}$. Compute the characteristic time τ.

[5 pt] ***e.*** Compute the rate of decrease of the kinetic energy of the rod per unit volume, and show that it equals the ohmic heating rate (dissipated electric power) per unit volume.

Problem 5.

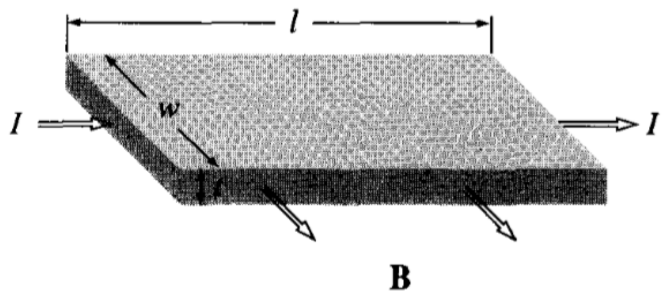
A current I flows to the right through a rectangular bar of conducting material, in the presence of a uniform magnetic field B pointing out of the page. Figure below.

[7 pt] ***a.*** If the moving charges are positive, in which direction are they deflected by the magnetic field? This deflection results in an accumulation of charge on the upper and lower surfaces of the bar, which in turn produces an electric force to counteract the magnetic one. Equilibrium occurs when the two exactly cancel. (This phenomenon is known as the Hall effect.)

[6 pt] ***b.*** Find the resulting potential difference (the Hall voltage),V_H, between the top and bottom of the bar, in terms of B, v (the speed of the charges), and the relevant dimensions of the bar.

[6 pt] ***c.*** A potential difference is applied between the ends of a strip of copper and a current of 100 A flows along its length. The strip is 20 cm long in the x-direction of a rectangular system of coordinates, 2 cm wide in the y- direction and 1 mm thick in the z-direction. A uniform magnetic field of 10 T is applied across the strip in the positive y- direction and the hall EMF is found to be 5 μV. Derive the magnitude and direction of the Hall field when the current flows in the positive x-direction.

[6 pt] ***d.*** Now for the experiment above, derive the concentration of free electrons, n.



Problem 6.

[8 pt] ***a.*** Define or state (be brief, but precise and define symbols and/or terms you use):

i. the Zeroth Law of Thermodynamics;

ii. temperature;

iii. absolute temperature;

iv. the First Law of Thermodynamics;

v. the thermodynamic entropy;

vi. the most general statement of the Second Law of Thermodynamics;

vii. the Third Law of Thermodynamics.

viii. the connection between absolute zero and the stopping of all molecular motion

In exact form, $N! = N^N e^{-N}\sqrt{2\pi N}$. In statistical mechanics, a very useful result when working with large numbers is Stirling's approximation: $\ln n! \approx n \ln n - n$.

[4 pt] ***b.*** Answer the following:

Guassian Integral

$$\int_{\infty}^{\infty} e^{-\alpha x^2} dx = ???$$

The multiplicity for an Einstein Solid

$$\Omega(N, q) = \binom{q+N-1}{q} = ???$$

[5 pt] ***c.*** Show that the multiplicity of an Einstein solid, for any large values of N and q , is approximately

$$\Omega(N, q) \approx \Big(\frac{q+N}{q}\Big)^q \Big(\frac{q+N}{N}\Big)^N \sqrt{\frac{N}{2\pi q(q+N)}}$$

[4 pt] ***d.*** Consider a monatomic ideal gas with a multiplicity given by

$$\Omega_N \approx \frac{1}{N!} \frac{V^N}{h^{3N}} \frac{\pi^{3N/2}}{(3N/2)!} \big(\sqrt{2mE}\big)^{3N}.$$

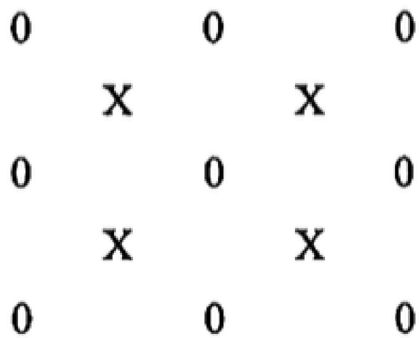
Show that the entropy of this gas is given by the Sackur-Tetrode equation:

$$S = Nk\Big[\ln\Big(\frac{V}{N}\Big(\frac{4\pi mE}{3Nh^2}\Big)^{3/2}\Big) + \frac{5}{2}\Big].$$

[4 pt] ***e.*** Compute the temperature $T(N, E)$. Compute the heat capacity $C(N, E)$. Compute the Helmholtz Free Energy $A(N, E)$.

Problem 7.

[25 pt] ***a.*** Consider a monatomic crystal consisting of N atoms. These may be situated in two kinds of positions:



a. Normal position indicated by 0.

b. Interstitial position, indicated by X.

Suppose that there is an equal number ($=N$) of both kinds of position, but that the energy of an atom at an interstitial position is larger by an amount ε than of an atom at a normal position. At $T = 0$ all atoms will therefore be in normal position. Show that, at a temperature T, the number n of atoms at interstitial sites is

$$n = Ne^{-\varepsilon/2kT}$$

for $n \ll N$.

Use the fact that the Helmholtz free energy is a minimum for equilibrium at constant volume and temperature.

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Problem 1.

Muons are short-lived particles that are produced in upper atmosphere (sat at a height of 5 km). The lifetime of a muon is 2.2 μs in its own rest frame. A muon is traveling at a speed of $v = 0.9c$.

[5 pt] ***a.*** What distance will it travel (in its own frame of reference) before decaying?

[5 pt] ***b.*** What is its lifetime according to the observer on the earth?

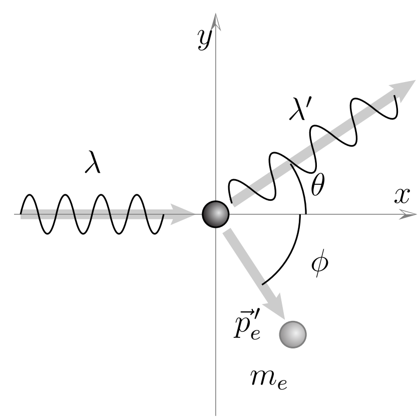
[5 pt] ***c.*** What is the distance travelled by muon before it decays according to the observer on earth?

[5 pt] ***d.*** According to the observer on the earth, the muon is created at a height of 5.0 km above the earth. What is this height according to the moving muon?

[5 pt] ***e.*** If an electron moves towards the earth at a speed of $0.5c$ relative to the earth, what is its speed relative to the muon? Assume that the electron?s motion is in the same direction as the muon.

Problem 2.

An X-ray photon of wave-length λ collides with an electron that was initially at rest. The photon deflects from its initial direction by an angle θ, while the electron recoils at an angle ϕ from the photon's initial direction.



[4 pt] ***a.*** Using the coordinate system and symbols given in the figure to the right, specify the energy and linear momentum of both the photon and the electron, before and after the collision, in terms of the quantities given in the figure and universal constants.

[6 pt] ***b.*** Write down the conservation equations, which completely determine all parameters in this scattering.

[7 pt] ***c.*** Derive Compton's formula, $\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)$ from the conservation equations you specified in the previous part.

[8 pt] ***d.*** An incoming neutrino ($m_\nu \approx 0$) of energy E_ν is analogously absorbed by a neutron, which emits an electron at the angle θ and thus turns into a proton. Using the same conservation laws, approximating $m_n \approx m_p + m_e$ and neglecting the proton recoil momentum but only when compared with $m_p c$, show that $|\vec{p}_e| \approx \sqrt{E_\nu(E_\nu + 2m_e c^2)}/c$.

Problem 3.

Consider a three dimensional anisotropic harmonic oscillator for which

$$H = \frac{\vec{p} \cdot \vec{p}}{2m} + 1/2\alpha x^2 + 1/2\beta y^2 + 1/2\gamma z^2$$

[5 pt] ***a.*** Define the raising and lowering operators for this Hamiltonian.

[5 pt] ***b.*** Express H in terms of raising and lowering operators.

[7 pt] ***c.*** What are the eigenvalues of H?

[8 pt] ***d.*** Write the time dependent Schrödinger equation for this system in the position-representation and in the momentum-representation.

Problem 4.

The wavefunction for the one dimensional infinite square-well is given by $\psi_n(x) = A\sin(n\pi x/L)$ where A is a normalization constant and $n = 1, 2, 3$, etc. .

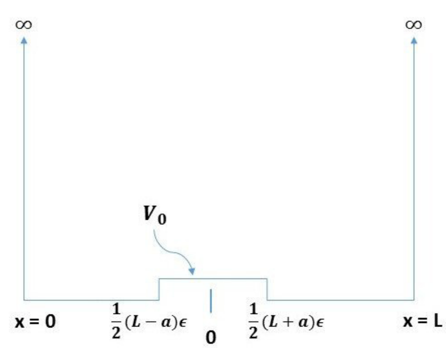
[5 pt] ***a.*** Normalize the wavefunction to find A.

[5 pt] ***b.*** Show that the quantized energy of a free particle in a one dimensional infinite square-well whose walls are separated by a length from $x \in [0, L]$ is $E_n = \frac{h^2\pi^2 n^2}{2mL^2}$.

[5 pt] ***c.*** Use Schrödinger equation to show that $\hat{H}\Psi_n = E_n\Psi_n$.

[5 pt] ***d.*** Calculate the expectation value of the energy, E.

[5 pt] ***e.*** A small step in the potential energy, V_0, is introduced in the one dimensional infinite square-well problem such that the potential energy is zero from $x = 0$ to $x = \frac{1}{2}(L-a)$, ϵ from $x = \frac{1}{2}(L-a)$ to $x = \frac{1}{2}(L+a)$, and zero from there to $x = L$. Find the first order correction to the energy of a particle confined to the well.



Problem 5.

The Pauli matrices given by: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

are linear operators of $\mathbb{C}^2$ to itself. In the Dirac formalism, the orthonormal basis vectors for $\mathbb{C}^2$ can be written as:

$$|v_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; , \; |v_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

[4 pt] ***a.*** Calculate the four matrix operators $|v_i\rangle \langle v_j|$ for i, j = 1, 2.

[6 pt] ***b.*** Use the results from (a) to write Pauli matrices (given above) as linear combinations of $|v_i\rangle \langle v_j|$.

[4 pt] ***c.*** What are the eigenvalues and eigenvectors of σ_3?

[6 pt] ***d.*** Write down the projection operators onto the subspace of $\mathbb{C}^2$ spanned by the eigenvectors of σ_3. Prove that the projection operators are idempotent ($p^2 = p$).

[5 pt] ***e.*** If a quantum mechanical system is described by the state:

$$|\Psi\rangle = \beta\,|v_1\rangle + \gamma\,|v_2\rangle$$

In a physical measurement, what is the probability of finding as a result the first and the second eigenvalue of σ_3?

Problem 6.

The quantum states available to a given physical system are (i) a group of g_1 equally likely states with a common energy value ε_1 and (ii) a group of g_2 equally likely states, with a common energy value ε_2. Show that the entropy of the system is given by

$$S = -k\big[p_1 \ln\big(p_1/g_1\big) + p_2 \ln\big(p_2/g_2\big)\big]$$

where p_1 and p_2 are, respectively, the probabilities of the system being in a state belonging to group 1 or to group 2: $p_1 + p_2 = 1$.

[13 pt] ***a.*** Assuming that the p's are given by a canonical distribution, show that

$$S = k\big[\ln g_1 + \ln\{1 + \big(g_2/g_1\big)e^{-x}\} + \frac{x}{1 + (g_1/g_2)e^{x}}\big].$$

where $x = (\varepsilon_2 - \varepsilon_1)/kT$, assumed positive.

[12 pt] ***b.*** Check that as $T \to 0, S \to k \ln g_1$. Interpret this result physically.

Problem 7.

Consider the linear harmonic oscillator (LHO) with mass M and the characteristic frequency ω.

[5 pt] ***a.*** Write down the Hamiltonian $\hat{H}$ in the coordinate representation, and then rewrite it using the operators $\hat{a} := \sqrt{\frac{m\omega}{2\hbar}}\big(\hat{x} + \frac{i}{m\omega}\hat{p}\big)$ and its conjugate $\hat{a}^\dagger$.

[5 pt] ***b.*** Denoting the eigenvectors of $\hat{H}$ by $|n\rangle$ with $|0\rangle$ the ground state, state: the energies (eigenvalues of $\hat{H}$), and the action of $\hat{a}$ and $\hat{a}^\dagger$ on $|n\rangle$.

[5 pt] ***c.*** Knowing that $\langle x|0\rangle = N\exp\{-\alpha x^2\}$: determine the constants α and N, and compute the wave-functions $\langle x|1\rangle$ and $\langle x|2\rangle$.

[5 pt] ***d.*** Modify the LHO by inserting an impenetrable wall at $x = 0$, and determine the spectrum (energy eigenvalues) and Hilbert space (complete set of eigenstates) for this reflecting oscillator.

[5 pt] ***e.*** Modify the LHO by adding $\lambda\,\delta(x)$ to the potential, assuming $\lambda > 0$. Prove that the energies of only the even-n states change to first two orders in stationary state perturbation theory.

Hint: *Consider and use the symmetries of $\psi_n(x) = \langle x|n\rangle$ with respect to the $x \to -x$ reflection.*