The Howard University
Department of Physics and Astronomy

Master of Science Comprehensive and Doctor of Philosophy Qualifying Exam

Written exam: Classical Physics

August 21, 2018

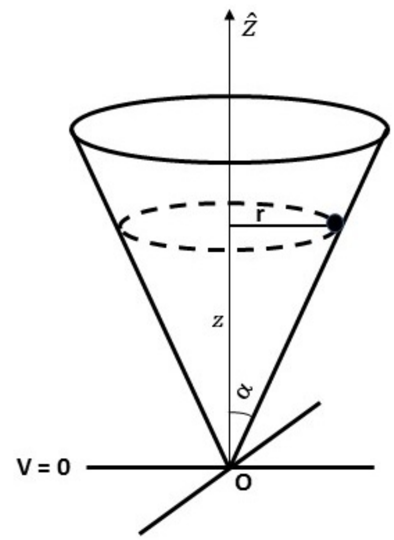
Attempt to solve ***four*** problems, at least one from each group. Circle the numbers below to indicate the problems you chose to solve:

1 2 3	4 5 6	7
Group A	Group B	Group C

1. Write in your chosen code-letter here: ☐
2. Write your code-letter and a page number (in sequential order) on the *top right-hand* corner of each submitted answer sheet.
3. Write only on one side of your answer sheets.
4. Start each problem on a new answer sheet.
5. Stack your answer sheets by problem and page number and then staple them (at the top *left-hand corner*), with this cover sheet on the top.

Problem 1.

A particle of mass moves on the inside surface of a smooth cone whose axis is vertical and whose half-angle is α as shown in the diagram.



[4 pt] ***a.*** Determine the kinetic energy, T, and potential energy, V, for the system.

[3 pt] ***b.*** Construct the Lagrangian function.

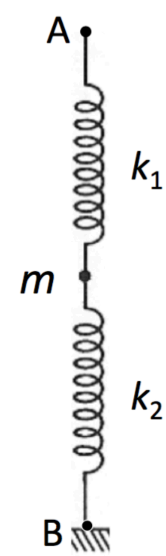
[6 pt] ***c.*** Find the Euler-Lagrange equation of motion.

[6 pt] ***d.*** θ is a cyclical coordinate for this system. Calculate an expression for the conserved quantity of angular momentum, ℓ, and then write an equation for $\dot{\theta}$ using the expression that you calculated for the angular momentum.

[6 pt] ***e.*** Find the period of small oscillations about a horizontal circular orbit located at a height h above the vertex.

Problem 2.

A particle of mass m can move in one dimension under the influence of two springs (see figure). The upper spring is attached to a moving point A that moves up with constant velocity v_0. The lower spring is attached to a fixed point B. The springs obey Hooke's law and have zero un-streched lengths and force constants k_1 and k_2 respectively.



[5 pt] ***a.*** Find the Lagrangian.

[5 pt] ***b.*** Find the Hamiltonian. Is it constant? (Justify your answer)

[5 pt] ***c.*** Find the energy. Is it conserved? (Justify your answer)

[5 pt] ***d.*** Obtain the equations of Hamilton.

[5 pt] ***e.*** Find the solution of the equations of motion.

Problem 3.

A yo-yo has a rotational inertia of I and a mass M. Its axle radius is R, and its string length is L. The yo-yo rolls from rest down to the end of the string. Neglect friction.

[5 pt] ***a.*** Compute the magnitude of its linear acceleration.

[5 pt] ***b.*** Compute the time it takes to reach the end of the string.

As the yo-yo reaches the end of the string, compute its:

[5 pt] ***c.*** linear speed;

[5 pt] ***d.*** translational kinetic energy;

[5 pt] ***e.*** rotational kinetic energy.

Problem 4.

A point charge q is placed at a distance R from the center of a grounded, perfectly conducting sphere of radius $a < R$. (Hint: the method of images may help.)

[6 pt] ***a.*** Compute the force that the so-charged sphere exerts on the point-charge q.

[6 pt] ***b.*** Show that for $R \gg a$, this force decreases as $1/R^3$, whereas when $a < R \ll 2a$, this force decreases as $1/(R-a)^2$.

[6 pt] ***c.*** Now re-consider the same point-charge q being brought to the same location near the conducting sphere, but with the conducting sphere now being electrically neutral and insulated. Recompute the force that it exerts on the point-charge q.

[7 pt] ***d.*** Compute the induced surface charge distribution on the surface of the conducting sphere.

Problem 5.

Consider a rectangular, perfect conductor wave-guide supporting the propagation of a monochromatic electromagnetic wave along its infinite length in the z-direction, with a width (in the x-direction) of a and a height (in the y-direction) of $b > a$.

[5 pt] ***a.*** State the boundary conditions on the components of $\vec{E}, \vec{B}$ at the walls, and in particular the conditions on E_z and B_z.

[8 pt] ***b.*** Starting from Maxwell's equations (there are no free charges of currents inside the wave-guide), obtain the wave equation which describes the $\vec{E}, \vec{B}$ fields of the lowest-frequency mode. (Hint: this mode has $\vec{E} = \hat{\mathbf{e}}_y E_y$.)

[6 pt] ***c.*** Determine the solutions for $\vec{E}, \vec{B}$ that satisfy all the boundary conditions.

[6 pt] ***d.*** For this lowest-frequency propagating mode, compute the dispersion relation, the phase velocity and the group velocity, and verify that $v_{\text{ph}} \cdot v_{\text{g}} = c^2$.

Problem 6.

[4 pt] ***a.*** Write down the four Maxwell's equations of electrodynamics, and state briefly (in a line or two) what each equation is expressing.

[8 pt] ***b.*** Assume a harmonic plane wave form for the electric and magnetic fields, such that each field has a constant amplitude. Also, assume that the fields to be monochromatic, with same (single) frequency, ω and same wave-verctor $\vec{k}$. Starting with the relevant Maxwell's equations, show that the dispersion relation is:

$$\vec{k} \cdot \vec{k} = \omega^2 \mu \epsilon. \tag{1}$$

For a dispersive material, assume that the permittivity and permeability are given as:

$$\epsilon = \epsilon_0\, n(\omega) \qquad \text{and} \qquad \mu = \mu_0\, n(\omega) \tag{2}$$

[5 pt] ***c.*** Starting with the relevant Maxwell's equations, find the phase and group velocity for the plane waves if the dispersion relation is given by the expression you derived in part ***b*** of this problem.

[4 pt] ***d.*** Show that if $\frac{dn}{d\omega} = 0$, there is no dispersion and the phase and group velocity always have the same sign.

[4 pt] ***e.*** When $n(\omega) < 0$, determine the condition for backwards traveling waves (i.e. the waves for which phase and group velocities have opposite signs). Such materials that exhibit negative indices of refraction are called negative-index (meta)materials.

Problem 7.

The molecules of a gas have only two possible states of internal energy with statistical weights (degeneracies) g_1 and g_2, and energies 0 and ε, respectively. Let n_1 and n_2 be the number of molecules in states *1* and *2*, respectively, so that $n_1 + n_2 = N =$ Avogadro's number.
[$k_B =$ Boltzmann's constant $= 1.38 \times 10^{-23}$ J/K]

[5 pt] ***a.*** Write the canonical partition function Z of the system.

[5 pt] ***b.*** Obtain the expression for the ratio n_2/n_1.

[5 pt] ***c.*** Solve for $n_1 = n_1(g_1, g_2, N, \varepsilon, T)$ and $n_2 = n_2(g_1, g_2, N, \varepsilon, T)$.

[5 pt] ***d.*** Derive an expression for the total internal energy, U, of the system.

[5 pt] ***e.*** Obtain an expression for the molar heat capacity at constant volume, C_V.

The Howard University
Department of Physics and Astronomy

Master of Science Comprehensive and Doctor of Philosophy Qualifying Exam

Written exam: Modern Physics

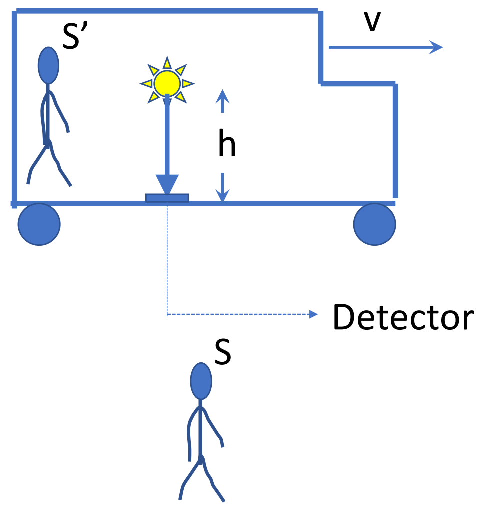
August 23, 2018

Attempt to solve ***four*** problems, at least one from each group. Circle the numbers below to indicate the problems you chose to solve:

1 2	3 4 5	6 7 8
Group A	Group B	Group C

1. Write in your chosen code-letter here: ☐
2. Write your code-letter and a page number (in sequential order) on the *top right-hand* corner of each submitted answer sheet.
3. Write only on one side of your answer sheets.
4. Start each problem on a new answer sheet.
5. Stack your answer sheets by problem and page number and then staple them (at the top *left-hand corner*), with this cover sheet on the top.

Problem 1.



Consider a van moving with relativistic velocity $\vec{v}$ (in x-direction for simplicity), with a light source inside the van at a height h from the floor of the van and a detector right below the source that records every time the light source is turned on and the signal (light) hits the detector. There are also two observers: S standing on the ground and S' in the van, who observe this event.

[4 pt] ***a.*** According to Special Relativity, what are the speeds of the signal from the light source as recorded by the ground observer, S and the observer, S'?

[4 pt] ***b.*** What are the distances (d) and (d') travelled by the signal as it travels between the source and the detector according to the ground observer, S and the observer, S'?

[6 pt] ***c.*** According to the ground observer, S and the observer, S', how much time elapses between turning on of the light signal and the light hitting the detector? Call these Δt and $\Delta t'$.

[11 pt] ***d.*** From the expressions found in previous part of this problem, derive the relation between Δt and $\Delta t'$? In your own words, explain the meaning of the relation. Which of the time intervals (Δt and $\Delta t'$) measured by the two observers is the proper time. Explain. *(5+3 +3 pts)*

Problem 2.

This problem concerns photoelectric and Compton effects.

[6 pt] ***a.*** Describe the photoelectric effect and give Einstein's explanation for the photoelectric effect.

[6 pt] ***b.*** Einstein's assumption about the nature of light was able to explain several experimental observations about the process. *In your own words and restricting yourself to 2-3 lines*, describe how these experimental observations were explained:

i. Within the limits of experimental accuracy, photoelectric effect is an instantaneous process.

ii. The photoelectrons come out with a range of kinetic energies, with a maximum value, K_{max}. This K_{max} does not depend on the light intensity!

iii. Higher the frequency, f, of the light (assuming $f > f_0$), greater is the energy of the photoelectrons.

Consider a metal plate made of iron that has threshold frequency of $f_0 = 1.1 \times 10^{15}$ Hz. Light of wavelength 250 nm is impinging on the metal.

[3 pt] ***c.*** Determine the work function of iron. Give answer in eV.

[5 pt] ***d.*** Determine the maximum kinetic energy possible for the photoelectrons. Give answer in eV.

[2 pt] ***e.*** Determine the stopping potential.

[3 pt] ***f.*** Compton effect in which X-ray photons are collided with electrons can be explained using the same assumptions about the nature of light as in the photoelectric effect. However, we ignore the binding energies of the electrons in the metal. Explain why this is a good approximation?

Problem 3.

A particle of mass moves in a two-dimensional infinite square well such that:

$$V(x) = \begin{cases} 0 & \text{for } \ 0 < x < L \quad \text{and} \quad 0 < y < L, \\ \to \infty & \text{at } \ x = 0, \quad x = L, \quad y = 0, \quad y = L. \end{cases} \tag{1}$$

[8 pt] ***a.*** Obtain the energy eigenvalues and eigenfunctions (you need not normalize them).

[8 pt] ***b.*** The potential well is deformed so that $L_x = L(1+\varepsilon)$ and $L_y = L(1-\varepsilon)$ where $0 < \varepsilon \ll 1$. What are the new energy eigenvalues and the corresponding eigenfunctions?

[9 pt] ***c.*** Discuss the effect of the deformation on the lowest three unperturbed levels by plotting the energy as a function of ε. Identify each level by the corresponding eigenfunction.

Problem 4.

Consider a simple quantum-mechanical system with two degrees of freedom, and an observable of this system represented by the hermitian matrix $\hat{F} = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right]$.

[4 pt] ***a.*** Determine the complex variables α, β so that the state vectors $|1\rangle = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]$ and $|2\rangle = \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]$ form an orthonormal basis for 2-dimensional vectors.

[6 pt] ***b.*** Construct the projection operators $\hat{\Pi}_n = |n\rangle\langle n|$, and than prove that: (i) $\sum_{n=1}^{2} \hat{\Pi}_n = \mathbb{1}$, and (ii) $\hat{\Pi}_1 \hat{\Pi}_2 = 0$.

[6 pt] ***c.*** Determine all possible results of (single attempts of) measuring $\hat{F}$, and the corresponding eigenvectors.

[3 pt] ***d.*** Calculate the expectation value of $\hat{F}$ in the pure state $|A\rangle = \frac{1}{\sqrt{5}}(|1\rangle - 2\,|2\rangle)$.

[6 pt] ***e.*** Calculate the probability that the measurement of the observable $\hat{F}$ would turn out to be $f_1 = 3$, if the system was initially prepared to be in the mixed state $\hat{\rho} = (\frac{1}{3}\hat{\Pi}_1 + \frac{2}{3}\hat{\Pi}_2)$.

Problem 5.

Consider a 2-dimensional isotropic harmonic oscillator constrained to the (x, y)-plane, that has the potential $V(x, y) = \frac{1}{2}m\omega^2(x^2 + y^2)$.

[6 pt] ***a.*** Write down the Schrödinger equation for stationary states with energies E_{n_x,n_y}, then rewrite it, using the standard operators $\hat{a}_x = \sqrt{\frac{m\omega}{2\hbar}}\big(x + \frac{i}{m\omega}\hat{P}_x\big)$ and $\hat{a}_y = \sqrt{\frac{m\omega}{2\hbar}}\big(y + \frac{i}{m\omega}\hat{P}_y\big)$, and their conjugates.

[6 pt] ***b.*** Determine the degeneracy of the energy levels from part ***a***, and show that it is a consequence of the rotational symmetry in the (x, y)-plane. (Hint: Consider the action of the mixed operator $\hat{L} = i\hbar(\hat{a}_x^\dagger\,\hat{a}_y - \hat{a}_y^\dagger\,\hat{a}_x)$ on the Hamiltonian, and draw conclusions from your result.)

[6 pt] ***c.*** Consider a perturbation that modifies the Hamiltonian by adding $\hat{H} \to \hat{H} + \lambda(x+y)$, where $\lambda > 0$ is a sufficiently small parameter. Prove that to 1st order, the energy levels remain the same, but that the states mix.

[7 pt] ***d.*** Compute the *exact* energy levels upon including the perturbation.

Problem 6.

Let $\vec{J}_1$ and $\vec{J}_2$ be the angular momenta operators for two independent systems and let $\vec{J} = \vec{J}_1 + \vec{J}_2$ be the total angular momentum (suppressing the operator sign "ˆ" on the $\vec{J}$'s here to avoid too many symbols).

[6 pt] ***a.*** State whether the following commutators are zero or non-zero, giving **reasons** for your answers (through direct computation or geometrical or physical reasoning). No points will be given for just an answer (even if correct) without a corresponding reason.

i. $[J_{1y},\ J_{2z}]$

ii. $[J_x,\ J_{2y}]$

iii. $[J_2^2,\ J_{2x} - J_{1x}]$

Now let angular momenta $j_1 = 1$ and $j_2 = 2$. Parts ***(b)-(e)*** of the problem concerns Clebsch-Gordan coeffcients, $\langle j_1, m_1, j_2, m_2 | j, m\rangle$.

[3 pt] ***b.*** Very briefly and precisely describe the significance (meaning) of Clebsch-Gordan coeffcients.

[2 pt] ***c.*** Determine the allowed values of total angular momentum j when we add angular momenta $\vec{J}_1$ and $\vec{J}_2$ with the particular eigenvalues j_1 and j_2.

[4 pt] ***d.*** Determine the the values of the Clebsch-Gordan coeffcients $\langle 1, -1, 2, -2 | 3, -3\rangle$ and $\langle 2, +1, 2, +2 | 3, +3\rangle$. Explain your answers.

[6 pt] ***e.*** Write the coupled (composite angular momentum) state $|j, m\rangle = |3, 2\rangle$ in terms of the uncoupled (product) basis states.

[4 pt] ***f.*** A large collection of mixed particles is prepared and this system is subjected to simultaneous measurement of the observables corresponding to the operators J^2 and J_z. The measurement yields $j = m = 1$ with a probability of $2/3$ and $j = m = 2$ with a probability of $1/3$. What was the state of the system (= beam of particles) just before the measurement?

Problem 7.

Consider a sample of N magnetic atoms, each with spin 1/2. The system is known to be ferromagnetic at very low temperatures; therefore as $T \to 0$ all the spins will be aligned. At sufficiently high temperatures, the spins will be randomly oriented. Neglect all other degrees of freedom, except for spin orientation.

[5 pt] ***a.*** Define the entropy S of the system in statistical terms, using $\Omega(n_i)$ = number of ways the system can be produced corresponding to the set of average number of particles n_i, where $i = 1, 2, \ldots$? (determine the range).

[5 pt] ***b.*** Obtain $S = S(n_1, n_2, N)$, where n_1 and n_2 are the number of atoms with spin up and down, respectively, and $N = n_1 + n_2$.

[5 pt] ***c.*** Define the entropy S of the system in terms of the specific heat, $C(T)$, and the (spin) temperature, T.

[5 pt] ***d.*** When $T \to 0$, obtain the values of n_1 and n_2. Also, determine the values of n_1 and n_2 when $T \to \infty$.

[5 pt] ***e.*** Derive an expression for $S = S(T)$ and obtain $S_\infty = S(T \to \infty)$.

Problem 8.

A quantum linear harmonic oscillator of mass m, frequency ω and electric charge q moves in the x-direction and is subject to a small, time-dependent perturbation to its Hamiltonian, $\hat{H} = \hat{H}_0 + \hat{H}'(t)$.

[4 pt] ***a.*** Expanding the time-dependent states $|\Psi(t)\rangle = \sum_n a_n(t)\, e^{-i\omega_n t}\, |n\rangle$ over the complete basis of stationary states, $\hat{H}_0\, |n\rangle = \hbar\omega(n+\frac{1}{2})\, |n\rangle$, show that $\omega_n = \omega(n+\frac{1}{2})$.

[5 pt] ***b.*** Expanding the time-dependent states $|\Psi(t)\rangle = \sum_n a_n(t)\, e^{-i\omega_n t}\, |n\rangle$ over the complete basis of stationary states, $\hat{H}_0\, |n\rangle = \hbar\omega(n+\frac{1}{2})\, |n\rangle$, show that $\dot{a}_{n'}(t) = \frac{1}{i\hbar} \sum_n \langle n'|\hat{H}'(t)|n\rangle\, e^{i\omega(n'-n)t}\, a_n(t)$.

[6 pt] ***c.*** By the time $t = 0$, the system is prepared to be in its ground state, and the perturbation is turned on: $\hat{H}'(t) = \lambda \hat{H}_1(t) \cdot \Theta(t)$, where $\Theta(t < 0) = 0$ and $\Theta(t \geqslant 0) = 1$. Expanding $a_n(t) = \sum_{k=0}^{\infty} \lambda^k a_n^{(k)}(t)$ for small λ, show that $a_n^{(1)}(T) = \frac{1}{i\hbar} \int_0^T \mathrm{d}t\; \langle n|\hat{H}'(t)|0\rangle\, e^{in\omega t}$, for $T > 0$.

[2 pt] ***d.*** The electric field $\vec{E}(t) = \hat{\mathbf{e}}_x\, E_0 \cos(\Omega t)\, \Theta(t)$ is turned on. Determine the perturbation $\hat{H}_1(t)$.

[5 pt] ***e.*** Compute (to lowest nonzero order in perturbation theory and ignoring the onset of any magnetic field) the transition amplitude $a_{0\to n}(T)$ and probability of the oscillator to be found in the n^{th} excited state after the finite amount of time T. (Hint: $\hat{x} = \sqrt{\hbar/2m\omega}(\hat{a}^\dagger + \hat{a})$ may be useful.)

[4 pt] ***f.*** Verify that your result is finite at the resonance frequency, $\Omega \to \omega$, and determine the condition on the magnitude of the perturbing electric field $\vec{E}$ for this result to be valid as a perturbative computation.