The Howard University
Department of Physics and Astronomy

Doctor of Philosophy Qualifying Exam

Written exam: January 26, 2023

Attempt to solve ***two*** problems, precisely one from each group. Circle the numbers below to indicate the problems you chose to solve:

1 2 3	4 5 6
Classical Mechanics	Electricity & Magnetism

1. Write in your chosen code-letter here:
2. Write your code-letter and a page number (in sequential order) on the *top right-hand* corner of each submitted answer sheet.
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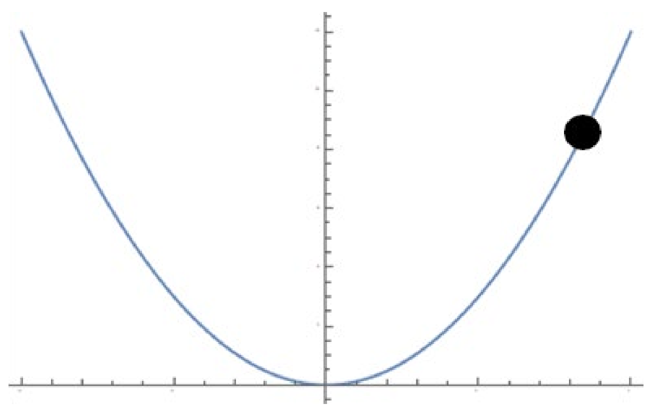
Part I

Classical Mechanics

Problem 1.

A ball of a point particle of mass M is constrained to move along a parabola shown in Figure **1**. The only forces to which the balls are subjected are the constraining forces that keep it on the parabola and the force of gravity.



[*4 pt*] **a.** If the ball starts at rest on the parabola with at x_0, find the velocity (vector) when it is at position x (assuming $|x| < |x_0|$).

[*3 pt*] **b**. Write the Lagrangian for the particle.

[*4 pt*] **c**. Find the equations of motion for particles (the equations of motion for just the x-coordinate are sufficient).

[*5 pt*] **d**. Estimate the period of the ball's oscillations if it is released from rest at a position x_0 close to the origin. State the conditions for which this estimate is accurate.

[*5 pt*] **e**. Estimate the correction to the period of oscillation to the next lowest order in x_0.

[*4 pt*] **f**. Assume the ball is starting from rest at horizontal position x_0. Find the force constraining the ball particle to move along in the parabola at horizontal position x (assuming $|x| < |x_0|$).

Problem 2:

Rockets are propelled by the momentum of the exhaust gases expelled from the tail. Since these gases arise from the reaction of the fuels carried in the rocket, the mass of the rocket is not constant, but decreases as the fuel is expended.

[*10 pts*] **a.** Show that the equation of motion for a rocket projected vertically upward in a uniform gravitational field, neglecting atmospheric friction, is

$$m\frac{dv}{dt} = -v'\frac{dm}{dt} - mg$$

where m is the mass of the rocket an v is the velocity of the escaping gases relative to the rocket.

[*5 pts*] **b.** Express Equation 1 as a velocity as a function of mass.

[*10 pts*] **c).** Integrate this equation to obtain v'as a function of m, assuming a constant rate of loss of mass. Show, for a rocket starting initially from rest, with v' equal to 2.1 km/s and a mass loss per second equal to 1/60th of the initial mass, that in order to reach the escape velocity, the ratio of the weight of the fuel to the weight of the empty rocket must be almost 300!

Problem 3.

Circular orbit under the influence of a central force:

[*8 pt*] **a**. Show that if a particle describes a circular orbit under the influence of an attractive central force directed at a point on the circle, then the force varies as the inverse fifth power of the distance.

[*7 pt*] **b**. Show that for the orbit described, the total energy of the particle is zero.

[*5 pt*] **c**. Find the period of the motion.

[*5 pt*] **d**. Find $\dot{x}$, $\dot{y}$, and v as a function of the angle around the circle and show that all three quantities are infinite as the particle goes through the center of force.

Part II

Electricity and Magnetism

Problem 4.

Two-point charges q and -q are located on the z-axis at z = +a and z = -a, respectively.

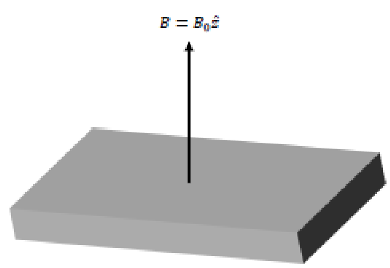
[*10 pt*] **a**. Find the electrostatic potential as an expansion in spherical harmonics and-powers of r for both r > a and r < a. (10 pints)

[*7 pt*] **b**. Keeping the product qa = (p/2) constant, take the limit of a → 0 and find the potential for $r \neq 0$. This is by definition, a dipole along the z-axis and its potential. (7 points)

[*8 pt*] **c**. Suppose now that the dipole of part b is surrounded by a grounded spherical shell of radius b concentric with the origin. By linear superposition, find the potential everywhere inside the shell. (8 points)

Problem 5.

A thick slab of metal is subjected to a magnetic field $B = B_0\,\hat{z}$ normal to the largest surface). There are n free carriers per unit volume in the metal with charge q, mass m_q. An external electric field $\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$ is applied to the slab. Charge Carriers in the metal are scattered at a rate of γ, which can be represented as an effective force. The mean velocity of an electron after it has been scattered is 0.



[*4 pt*] **a**. If $B_0 = 0$ the applied electric field is constant as a function of time, show that the mean velocity of a charge carrier is given by $\langle \overrightarrow{v_{e^-}} \rangle = \frac{q}{m_q \gamma}\left(E_x\hat{x} + E_y\hat{y} + E_z\hat{z}\right)$

[*6 pt*] b. Find the mean velocities in the x- and y-directions of a charge carrier for $B_0 \neq 0$.

[*5 pt*] **c**. Now, assume that the driving electric fields are now oscillating with frequency ω. You may use $E_x(t) = E_x(0)e^{-i\omega t}$, and $E_y(t) = E_y(0)e^{-i\omega t}$, $E_z = E_z(0)e^{-i\omega t}$. Find the mean values of v_x, v_y, and v_z at time t as a function of E_x, E_y, q, m, γ, ω, and B_0.

[*5 pt*] **d**. How can your answer from (c) be used to calculate the conductivity tensor [σ]? If j = [σ] $\vec{E}$ (where $\vec{j}$ is the current density)? What does it mean to have non-zero off-diagonal elements? Show that the tensor takes the form:

$$[\sigma] = \begin{bmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

[*5 pt*] **e**. What do the eigenvectors of a conductivity tensor mean (mathematically and physically)? What does a complex eigenvalue mean in this context? (5 points)

Problem 6.

The electrostatic energy of a charge distribution is given by

$$U = (1/2) \iiint \rho(\vec{r}) V(\vec{r})\, dxdydz \qquad (1)$$

where $\rho(\vec{r})$ charge density at the position $\vec{r}$ and $V(\vec{r})$ is the potential.

[*4 pt*] **a**. What assumptions are necessary to go from Equation (1) to Equation (2) below?

$$U = (1/2) \iiint \left|\vec{E}(\vec{r})\right|^2 dxdydz \qquad (2)$$

[*2 pt*] **b**. If you use Equation (2) to solve for the electrostatic potential energy between two charged point particles, why do you get a different answer than if you used Equation (1)?

[*2 pt*] **c**. If you place a dielectric or metal inside of an electric field, does the total electrostatic energy increase or decrease? Explain why (you may limit your answer to 1-3 sentences).

[10 *pt*] **d**. You begin with a charge distribution $\rho_1(\vec{r})$ over all of the space whose electrostatic energy is U_1. What will be the energy of the charge distribution if the charge distribution became n times less dense? Prove that your answer holds for an arbitrary $\rho_1(\vec{r})$. Express your answer in terms of U_1 and n.

[*3 pt*] **e**. If all of the charges in the distribution for part (d) have the same sign (i.e. all are positive or all are negative), explain why a more sparse distribution has a lower potential energy.

[*4 pt*] **f**. Why is the electrostatic energy of a charge distribution in a conductor minimized when all of the charges are on the surface of the conductor?

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Quantum Mechanics	Statistical Mechanics

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Part III

Quantum Mechanics

Problem 7.

A non-relativistic quantum particle of mass m is constrained to move freely along the x-axis between impenetrable walls placed at x = 0 and x = L.

[*7 pt*] **a**. Write down the Schrödinger equation, compute the complete set of normalized stationary states and the corresponding energy eigenvalues.

[*5 pt*] **b.** Assume now that the wall at x = L is very slowly (so that the quantum state of the particle adapts continuously) moved out to x = 2L. Determine the new energy levels.

[*7 pt*] **c.** Assume now that the particle is in the ground state when the wall at x = L is instantaneously moved to x = 2L. Calculate the probability that the particle is also in the ground state of the stretched system.

[6 *pt*] **d.** The original particle, in the original box, is in the n^{th} state up to the time t = 0, when it is subject to a perturbation $H' = Axe^{-(t/\tau)}2$, for all t > 0. Calculate the probability that it will be in another, k^{th} state by the time t = +∞.

Hints: You may find the integrals: $\int_0^\pi d\varphi\; Sin(k\varphi)Sin(n\varphi) = (\frac{\pi}{2})^2\,\delta_{k,n}$ - $[1-\delta_{k,n}]\frac{2kn}{(k^2-n^2)^2}[1 - (-1)^{k+2}]$, $\int_0^\pi d\varphi\; Sin(k\varphi)Sin(n\varphi) = (\frac{\pi}{2})^2\,\delta_{k,n}$ and $\int_{-\infty}^{+\infty} dz\; e^{-z^2}\,\sqrt{\pi}$ useful.

Problem 8.

Consider a standard one-dimensional square well:

$$V(x) = \begin{pmatrix} 0, & |x| > a \\ -V_0, & |x| < a \end{pmatrix}$$

Particles of energy E > 0 are incident on it from the left.

[*15 pt*] **a.** Calculate the transmission coefficient T.

[*5 pt*] **b.** How does T behave for very large energies?

[*5 pt*] **c.** What is its low-energy limit?

Problem 9.

Let $A\hat{}$ and $B\hat{}$ be linear operators, and let $\hat{C}$ denote their commutator, *i.e.*

$$\hat{C} \equiv [\hat{A}, \hat{B}]$$

[*7 pt*] **a.** Show that $\hat{C}$ is also a linear operator.

[*8 pt*] **b.** Suppose $\hat{A}$ and $\hat{B}$ share a common eigenfunction, φ_{ab}, *i.e.*

$$\hat{A}\varphi_{ab} = a\varphi_{ab} \text{ and } \hat{B}\varphi_{ab} = b\varphi_{ab}\,.$$

Show that φ_{ab} must be annihilated by the commutator, *i.e.*

$$\hat{C}\varphi_{ab} = 0.$$

[*5 pt*] **c.** Suppose $\left[\hat{A}, \hat{B}\right] = 0$. Such operators are said to *commute*. Can $\hat{A}$ and $\hat{B}$ share common eigenfunctions?

[*5 pt*] **d.** Suppose $\hat{A}$ *and* $\hat{B}$ commute. Are all eigenfunctions of $\hat{A}$ necessarily also eigen-functions of $\hat{B}$? If so, explain why; if not, give a simple counterexample.

Part IV

Statistical Mechanics

Problem 10.

Consider a rigid lattice of distinguishable spin½ atoms in a magnetic field H. The spins have two states, with energies $-\mu_o H$ and $+\mu_o H$ for the spin up (↑) and spin down (↓), respectively, relative to H. The system is at temperature T.

[*10 pt*] **a.** Calculate the canonical partition function Z for this system.

[*10 pt*] **b.** Determine the total magnetic moment $M = \mu_0 (N_\uparrow - N_\downarrow)$ of the system.

[*5 pt*] **c.** Calculate the entropy, S, of the system.

Problem 11.

The binary alloy: A binary alloy (as in β brass) consists of N_A atoms of type A, and N_B atoms of type B. The atoms form a simple cubic lattice, each interacting only with its six nearest neighbors. Assume an attractive energy of $-J$ ($J > 0$) between like neighbors $A - A$ and $B - B$, but a repulsive energy of $+J$ for an $A - B$ pair.

[*10 pt*] **a.** What is the minimum energy configuration, or the state of the system at zero temperature?

[*10 pt*] **b.** Estimate the total interaction energy assuming that the atoms are randomly distributed among the N sites; i.e. each site is occupied independently with probabilities $p_A = N_A/N$ and $p_B = N_B/N$.

[*5 pt*] **c.** Estimate the mixing entropy of the alloy with the same approximation. Assume $N_A, N_B \gg 1$.

Problem 12.

Consider an adsorbent surface having N sites, each of which can adsorb one gas molecule. This surface is in contact with an ideal gas with chemical potential μ (determined by pressure p and temperature T). Assume that the adsorbed molecule has energy $-\varepsilon_0$ compared to one in a free state.

[10 *pt*] **a.** Calculate the canonical (Z) and grand canonical (Q) partition functions for the system.

[*5 pt*] **b.** Obtain the average number of particles, N_{avg}.

[10 *pt*] **c.** Determine the coverage ratio $\theta = N_{avg}/N$ as a function of μ, ε_0, and T. Note that the coverage ratio provides the ratio of adsorbed molecules to adsorbing sites on the surface.